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A SIMULTANEOUS EQUATION MODEL TO DETERMINE TAPER CURVE

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SELOSTE:

RUNKOKÄYRÄN MÄÄRITTÄMINEN SIMULTAANISEN MONIYHTÄLÖMALLIN AVULLA

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A simultaneous-equation model to determine the taper curve is presented. The diameters at relative heights are endogeneous variables and height an exogeneous variable. Any equation may be substituted by the measured value of the diameter. Solution of the system of equations yields 11 diameters at relative heights. Intermediate values are obtained by interpolation. Interpolation allows the use of diameters measured at absolute heights, too.

INTRODUCTION

Generally, taper curve equations yield diameters for the whole stem or for a segment of the stem (e.g. MAX and BURKHART 1976). However, it is evident that equations for single diameters are less complicated than taper equations for a relatively long segment of the bole (KUUSELA 1965). If a sufficiently large number of stem diameters is known the rest of the diameters may be estimated by interpolation. Estimation of the tree taper is also markedly simplified if diameters at relative heights instead of absolute heights ere employed (PRODAN 1951).

In this paper the tree taper is determined by a system of equations in which the number of equations is the same as the number of diameters to be estimated. Three possibilities are available for the formulation of the system of equations:

1. A single-equation model. Each diameter is estimated by one equation not related to the other equations.
2. A multi-equation recursive model. The equations are ordered so that the diameter estimated from the first equation may be substituted in the second equation etc.
3. A simultaneous-equation model. Each diameter is described by an equation in which all other diameters (other endogeneous variables) in addition to the other characteristics of the tree and its environment (exogeneous variables) are explanatory variables. Solution of the system of equations yields the diameters.

If linear relationships prevail between the diameters it can be proved that the single-equation models and the simultaneous-equation model yield exactly the same results. However, the single-equation models require a new set of equations whenever the measured diameters are changed. The simultaneous-equation model, on the other hand, is valid for any combination of measured diameters.

This paper presents a simultaneous-equation model which yields 11 diameters at relative heights of the tree. Application of the model is also demonstrated.

DATA

Data comprised 135 Scots Pines felled from several stands in Central Finland. Dbh varied from 75 to 304 millimeters and height from 7.5 to 25.3 meters. Dbh, height, stump diameter, diameter at 0.55 meter's height and diameters at 1 meter's intervals for trees shorter than 12 meters and diameters at 2 meter's intervals for taller trees were measured. The taper curve for each tree was derived by paired second-degree equations passing through four successive diameters. Diameters $d_{.0h}$, $d_{.05h}$, $d_{.1h}$, $d_{.2h}$, $d_{.3h}$, $d_{.4h}$, $d_{.5h}$, $d_{.6h}$, $d_{.7h}$, $d_{.8h}$, and $d_{.9h}$ were estimated from the taper curve. The correlations between the diameters are given in Table 1.

Table 1. Correlations between the diameters.

Taulukko 1. Läpimittojen väliset korrelaatiokertoimet.

	$d_{.05h}$	$d_{.1h}$	$d_{.2h}$	$d_{.3h}$	$d_{.4h}$	$d_{.5h}$	$d_{.6h}$	$d_{.7h}$	$d_{.8h}$	$d_{.9h}$
$d_{.0h}$	.964	.965	.966	.962	.962	.959	.950	.928	.894	.848
$d_{.05h}$		.994	.985	.983	.979	.976	.968	.946	.915	.855
$d_{.1h}$			.994	.992	.989	.984	.976	.955	.925	.872
$d_{.2h}$				.998	.996	.992	.985	.967	.940	.886
$d_{.3h}$					.998	.994	.988	.971	.943	.890
$d_{.4h}$						.997	.991	.974	.946	.891
$d_{.5h}$							.995	.981	.954	.895
$d_{.6h}$								.991	.967	.910
$d_{.7h}$									.985	.921
$d_{.8h}$										.952

ESTIMATION OF THE PARAMETERS

The parameters of the equations which yielded the diameters were derived by the least-squares method. The independent variables included the other diameters at relative heights and height of the tree. No transformations were applied to the diameters. The height was squared, too. The regression coefficients, coefficients of determination and standard errors of estimates are given in Table 2.

Table 2. Regression coefficients, coefficients of determination, and standard errors of estimates for the equations of the simultaneous-equation model.

Taulukko 2. Simultaanisen moniyhtälömallin yhtälöiden regressiokertoimet, yhteiskorrelaatiokertoimet ja keskivirheet.

Independent variables *Selittävät muuttujat*	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
	Dependent variable — *Selitettävä muuttuja*										
	$d_{.0h}$	$d_{.05h}$	$d_{.1h}$	$d_{.2h}$	$d_{.3h}$	$d_{.4h}$	$d_{.5h}$	$d_{.6h}$	$d_{.7h}$	$d_{.8h}$	$d_{.9h}$
$d_{.0h}$		.0745	−.0158	.0312	−.0154	.0036	.0075	−.0069	.0107	−.0230	.0345
$d_{.05h}$	.8978		.4782	−.0924	.0056	−.0116	−.0173	.0433	−.0236	.0595	−.1250
$d_{.1h}$	−.5290	1.3329		.3460	.0810	−.0547	.0667	−.0441	−.0202	−.0862	.1177
$d_{.2h}$	1.4441	−.3543	.4762		.3567	.0638	−.0463	.0608	−.0893	.1157	−.1657
$d_{.3h}$	−1.2487	.0376	.1950	.6241		.4961	−.2347	.0598	.0436	.0361	.1172
$d_{.4h}$	.4650	−.1253	−.2113	.1791	.7956		.7806	−.3121	.2047	−.3556	.2759
$d_{.5h}$	.8330	−.1597	.2212	−.1116	−.3234	.6709		.7391	−.3350	.4240	−.5030
$d_{.6h}$	−.5749	.3014	−.1100	.1102	.0620	−.2017	.5559		.7370	−.4463	.5490
$d_{.7h}$	.7552	−.1391	−.0426	−.1370	.0382	.1119	−.2131	.6231		.8250	−.6909
$d_{.8h}$	−1.4289	.3073	−.1598	.1558	.0278	−.1707	.2369	−.3315	.7247		.9790
$d_{.9h}$	1.0903	−.3276	.1107	−.1133	.0458	.0672	−.1426	.2069	−.3080	.4968	
const.	−8.8869	11.1760	−9.1997	6.8207	−.8700	.3077	−1.4939	1.5149	−2.6503	1.6093	3.5759
h	.7870	−1.2206	1.4848	−.9924	−.0049	.0265	.1574	−.2634	.4043	−.1354	−.1344
h^2	−.0326	.0211	.0292	.0323	−.0070	.0042	−.0128	.0133	−.0143	.0146	.0097
R^2	.943	.991	.997	.997	.998	.999	.998	.997	.995	.990	.938
s_d, mm	15.4	4.4	2.7	2.3	1.7	1.4	1.5	1.7	1.8	2.0	2.7

INTERPOLATION OF THE INTERMEDIATE VALUES

To complete the taper curve model an interpolation method is needed to derive the intermediate values. In this paper the following formula is suggested:

$$d_h = f2 + (f1-f2)\,\frac{|b2|}{|b1|+|b2|}\left(\frac{h3-h}{h3-h2}\right)^m + (f3-f2)\,\frac{|b2|}{|b2|+|b3|}\left(\frac{h-h2}{h3-h2}\right)^n \qquad (12)$$

$h2 \leq h \leq h3$

The variables in the formula are explained in Figure 1.

The formula ensures that there is no change in the first derivatives of the taper curve in the joints and that the taper curve segment equals a straight line whenever there is no tapering between two successive diameters. For the first and last intervals the missing values for b1 and b3 may be estimated from b2.

The values of the parameters m and n should be determined from proper data. Also the possible dependence of these parameters upon the relative height should be studied. In the following the value 2.4 was applied for both of the parameters.

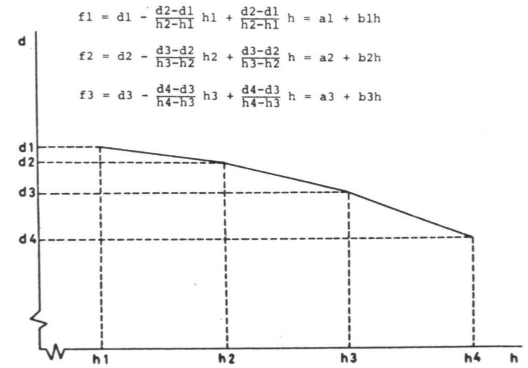

Figure 1. Calculation of straight lines connecting successive diameters

Kuva 1. Peräkkäisiä läpimittoja yhdistävien suorien laskenta

APPLICATION OF THE MODEL

If one or more diameters at prefixed relative heights are measured the application of the model is very easy.

Example 1. $d_{.1h} = 181$ mm
$d_{.3h} = 154$ mm
$h = 19.0$ m

Equations (3) and (5) are substituted by equations $d_{.1h} = 181$ and $d_{.3h} = 154$. The solution of the new system of equations is

$d_{.0h} = 243$
$d_{.05h} = 194$
$d_{.1h} = 181$
$d_{.2h} = 165$
$d_{.3h} = 154$
$d_{.4h} = 144$
$d_{.5h} = 132$
$d_{.6h} = 117$
$d_{.7h} = 99$
$d_{.8h} = 75$
$d_{.9h} = 43$

If the diameters are measured at arbitrary heights the solution of the system of equations becomes more complicated.

Example 2. dbh = 186 mm
$h = 19.0$ m

First, the original system of equations is solved. The solution is

$d_{.0h} = 282$
$d_{.05h} = 220$
$d_{.1h} = 207$
$d_{.2h} = 192$
$d_{.3h} = 180$
$d_{.4h} = 169$
$d_{.5h} = 154$
$d_{.6h} = 136$
$d_{.7h} = 113$
$d_{.8h} = 84$
$d_{.9h} = 48$

Dbh is estimated by formula (12). This estimated dbh equals 214 mm. A new value is calculated for $d_{.05h}$ by multiplying the original value by the ratio of the measured and estimated dbh's. Equation (2) is substituted by equation $d_{.05h} = 220 \cdot 186/214 = 191$. The new system of equations is solved and the iteration procedure is continued until the estimated and measured dbh's are equal. The final solution is

$$d_{.0h} = 244$$
$$d_{.05h} = 191$$
$$d_{.1h} = 180$$
$$d_{.2h} = 168$$
$$d_{.3h} = 158$$
$$d_{.4h} = 148$$
$$d_{.5h} = 135$$
$$d_{.6h} = 121$$
$$d_{.7h} = 102$$
$$d_{.8h} = 77$$
$$d_{.9h} = 45$$

If two or more diameters at arbitrary heights are measured the preceding procedure is applied to each diameter closest to the measured diameter.

RELIABILITY OF THE MODEL

The simultaneous equation model was tested employing the same 135 trees upon which the model was based. In the first test it was assumed that only dbh and height are known. In the second test it was assumed that d_6 is also known. Both absolute and relative standard errors and biases of the diameters are presented in Table 3. Slight biases in the absolute values are due to the inadequacy of the interpolation formula (12) to describe the taper of the segments of the tree.

The volumes were calculated by intergrating the taper curve. To avoid negative bias the expected value of each diameter($\hat{d}$) was replaced by $\sqrt{\hat{d}^2 + s_{\hat{d}}^2}$. If only dbh and height were known the relative standard error of the volume estimate was 7.0 per cent and the relative bias +0.6 per cent. When d_6 was added the relative standard error of the volume estimated was 2.8 per cent and the relative bias +0.3 per cent.

Table 3. Standard errors (s) and biases (b) of the diameters.

Taulukko 3. Läpimittojen keskivirheet (s) ja harhat (b).

Relative height — *Suhteellinen korkeus*	Measurements — *Mittaukset*							
	$d_{1.3}$, h				$d_{1.3}$, d_6, h			
	mm		per cent, %		mm		per cent, %	
	s	b	s	b	s	b	s	b
0.0	17.0	−.0	8.1	+.7	16.0	+.0	7.7	+.7
0.05	4.4	−.0	3.1	+.2	4.6	−.0	3.2	+.1
0.1	2.3	−.0	1.5	+.1	1.7	+.0	1.4	+.1
0.2	5.2	−.0	3.5	+.2	2.9	+.1	2.2	+.2
0.3	5.6	−.0	4.1	+.2	2.2	+.1	2.0	+.2
0.4	6.1	−.0	4.7	+.2	2.0	+.1	1.8	+.1
0.5	6.3	−.0	5.4	+.3	2.8	+.0	2.1	+.1
0.6	6.4	−.0	6.6	+.4	3.6	−.0	3.4	+.2
0.7	6.9	−.0	8.6	+.6	4.9	+.0	5.4	+.3
0.8	6.0	−.0	10.6	+1.0	4.8	+.0	8.0	+.6
0.9	4.4	−.0	13.2	+1.7	4.0	+.0	11.9	+1.5

DISCUSSION

The taper curve model presented in this paper serves mainly to illustrate the potentials of the simultaneous equation approach in the estimation of the taper curve. Data are impaired by a number of weaknesses. The trees do not represent any relevant parent population. Diameters were not measured at relative heights but estimated from the measurements made at absolute heights. The accuracy of the measurements may also have left something to be desired.

Analysis of the residuals of the regression equations did not indicate any serious nonlinearities between the relative-height diameters. Possible non-linearities would make the use of the model less convenient and increase the computer time as iteration has to be employed in the solution of the system of equations.

The height was taken into the model mainly to demonstrate the possibility to use exogeneous variables. The covariation between the relative-height diameters and the height of the tree may as well be due to the covariation between the height and stand characteristics. With more representative data, stand characteristics might prove to be useful as exogeneous variables. Dbh would have improved the model but it was not used in order to maintain the generality of the model.

Another open question is the optimal number and the optimal measurement heights of the diameters to be taken into the system.

The main advantage of the taper curve model described in this paper is the complete freedom in the choice of the number of diameter measurements and measurement heights.

LITERATURE CITED

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SELOSTE:

RUNKOKÄYRÄN MÄÄRITTÄMINEN SIMULTAANISEN MONIYHTÄLÖMALLIN AVULLA

Tutkimuksessa esitetään männylle laskettu simultaaninen moniyhtälömalli (taulukko 2), jonka ratkaisusta saadaan 11 läpimittaa puun suhteellisilta osakorkeuksilta. Näiden läpimittojen väliarvojen laskemiseksi annetaan kaava (12). Esimerkein osoitetaan, kuinka simultaanisen moniyhtälömallin halutut yhtälöt voidaan korvata joko suoraan suhteellisilta osakorkeuksilta mitatuilla läpimitoilla tai absoluuttisilta osakorkeuksilta mitatuista läpimitoista johdetuilla suhteellisten osakorkeuksien läpimitoilla. Mallilla saatu runkokäyrä kulkee kaikkien mitattujen läpimittojen kautta, eikä malli aseta mitään vaatimuksia läpimittojen mittauskorkeuksille tai lukumäärälle.