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MÄNNYN RUNKOKÄYRÄMALLEJA JA NIIDEN SOVELLUTUKSIA

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PREFACE

This study was jointly planned by the two authors. Varmola led the field work, prepared and tested the data, made the preliminary calculations, prepared the final version of the computer programs, and wrote the Finnish summary. Kilkki was in charge of the main part of the calculations and wrote the English text.

Professor Hannu Väliaho, Professor Yrjö Vuokila, and Mr Jouko Laasasenaho read the manuscript and made useful comments on it. Discussions with Messrs Risto Ojansuu, Risto Päivinen, Juha Puranen, and Markku Siitonen contributed to better understanding of the problem. The field work was carried out by Messrs Ville Hallikainen and Ossi Huhtala. Mr Ashley Selby checked the English text and Ms Helena Henttonen drew the figures.

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Helsinki March, 1981

Pekka Kilkki Matti Varmola

CONTENTS

1. INTRODUCTION

The use of simultaneous equations for the determination of taper curves was introduced in the previous papers (KILKKI et al. 1978; KILKKI and VARMOLA 1979). In these models each relative-height diameter was predicted by the other relative-height diameters (endogenous variables) and by the height of the tree (exogenous variable). In the first paper, the simultaneous equation model was linear with respect to the diameters and in the second paper nonlinear. Both models were nonlinear with respect to height.

The applicability of the simultaneous models to the determination of the taper curves is based upon two factors. The first factor is the generality of the taper curve model based upon the simultaneous equations. Any endogenous variable in the model can be exogenized through its substitution by a measured value. Thus the simultaneous model, in fact, comprises a large number of separate taper curve models with different sets of independent (measured) variables. The difficulty that the measured diameters may not coincide with the endogenous variables of the simultaneous model can be solved by the employment of interpolation formulas.

The other factor that facilitates the use of the simultaneous models is the relatively simple form of the separate equations. The wellknown complexity of taper curve models manifests itself through the interaction between the equations. It is also to be noticed that least squares method is valid in the estimation of the model parameters.

Both of the previous studies were handicapped by inappropriate data. In the first study, the data comprised only 132 trees and the diameter measurements were not originally made at the relative heights but later interpolated from the absolute-height diameter measurements. In the second study, the data comprised only small trees. In this paper we try to test the validity of the simultaneous equation model approach with more comprehensive data.

The aim of this study is to develop simultaneous equation taper curve models for Scots pine and to demonstrate the application of these models in the estimation of the volume and its error variance for various parts of the stem.

2. DATA

The parent population was all Scots pines in Southern and Central Finland. It was decided to measure a sample of 500 trees. Because of the limited time and financial resources it was not possible to measure a sample which would fulfill the most rigorous requirements of representativeness. Consequently, the sample trees were taken from stands marked for cutting where it was possible to fell the trees before measurements.

Trees from 29 stands were measured. The location of the stands and the number of the sample trees per stand are given in Fig. 1. Only stands on mineral soils were accepted. The sites varied from *Oxalis-Myrtillus* site type to *Calluna* site type (see CAJANDER 1949), the majority being measured from *Vaccinium* site type. The stoniness of the soils varied from almost stoneless to extremely stony. Some of the sites were boggy. The taxation class varied from IA to IV (see KUUSELA 1978). Most of the stands had been established by natural regeneration. Only 5 stands had been established artificially. The majority of the stands were mature, but some young stands were included in the data.

Since it is important that the sample trees represent all kinds of trees, the measurements were largely concentrated on areas to be clear cut. In order to obtain a sufficient number of young and small sized trees, construction cleanings were used in addition to the mature stands which are normally the objects of clear cutting. In thinning stands, the measurements were concentrated on skidding roads. In some cases the felling of unmarked trees was also allowed.

A maximum number of sample trees was established for each 2 cm – 2 m $d_{1.3}$-h class. This number was originally set to 4, but it was later increased to 5. Because of the estimation errors, the final number of the sample trees per class was in several cases 6 and in one case 7 (see Table 1).

In order to avoid the concentration of the trees of one stand into very few $d_{1.3}$-h classes, only one tree from each stand was accepted into a class. A maximum number of 20 trees per stand was measured. The homogeneity of the stands and the earlier fulfilled quotas of the most usual $d_{1.3}$-h combinations reduced this number to less than 10 sample trees in some stands. The total number of sample trees was 495. Because of measurement errors and/or abnormalities 3 trees were disqualified. The final size of the sample was thus 492 trees (Table 1).

The sampling of the trees was not performed by any pure systematic or random sampling procedure. Nevertheless, it is improbable that any subjective bias in the sample tree selection occurred. The sampling

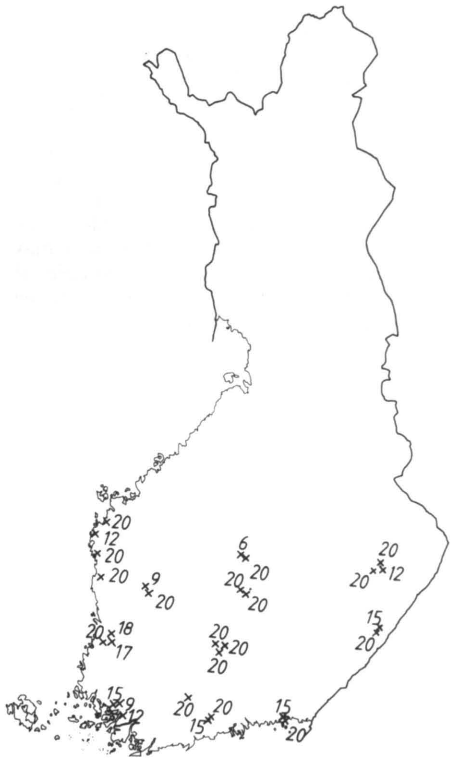

Fig. 1. Location of the sample stands. The figures indicate the number of sample trees per stand.

Kuva 1. Koepuumetsiköiden sijainti. Numerot ilmaisevat metsiköstä mitattujen koepuiden lukumäärän.

Table 1. Breast height diameter and height distribution of the sample trees.

Taulukko 1. Koepuiden jakautuminen rinnankorkeusläpimitta- ja pituusluokkiin.

$d_{1.3}$,	h, m																
cm	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33
1																	
3		4	4	1													
5		2	5	5	3												
7		2	3	6	3	3	1										
9			2	5	4	5	3										
11			1	3	5	5	5	1									
13		1		4	5	5	4	3									
15				1	5	4	6	3	2	2							
17				3	3	5	5	4	5	2	1						
19				1	1	2	6	5	5	1	2	1					
21						4	4	5	5	5	2		1				
23			1			2	6	7	5	4	4	2					
25					1	3	2	5	5	4	5	5	1				
27			1		1		1	5	5	5	4	1	3	1			
29							2	4	6	4	4	4	3	2			
31							2	5	5	6	5	3	4	1			
33								1	4	5	3	4	4	3			
35						1		3	5	4	5	4	2	2	1		
37								2	3	1	3	4	4	2	1	1	
39							1	1	1	1	3	1	2	2		1	
41								2	1	2	1	1	4	4	2	1	
43										1			1	1	1	1	
45											1	3	2	3	3		
47													3		2		
49													2	2	1	1	1
51														1	2	1	
53																	
55													1	1			
57																1	

was started from one end of the stand and it proceeded in a normal tree tally order. Quite frequently it took a large number of $d_{1.3}$-h measurements to find a tree that qualified for a sample tree.

Only one-topped living trees were accepted. The breast height of the tree was marked before felling. The height of the felled tree was measured to an accuracy of one centimeter. The crown height (h_C) was measured to an accuracy of 10 centimeters. The crown height indicates the point where the live crown begins. A single living branch separated by more than two dead branch whorls was not counted to the crown. The age of the tree was measured as the number of annual rings in the stump.

The diameters of the felled trees were measured in two opposite directions for all trees at the breast height and for trees taller than 7 meters also at six meters' height. Similar measurements were made at the following relative heights: 1, 2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90, and 95 per cent of the height of the tree. The diameter measurements were made to an accuracy of one millimeter. If the measurement height coincided with a branch or knot bump, the measurement height was moved to the closest

Table 2. Minima, maxima, and means of the sample tree characteristics.
Taulukko 2. Koepuutunnusten minimit, maksimit ja keskiarvot.

Variable *Muuttuja*	Minimum *Minimi*	Maximum *Maksimi*	Mean *Keskiarvo*
$d_{1.3}$	26.0 mm	569.5 mm	247.1 mm
d_6	0 "	466.0 "	
$d_{.01}h$	39.5 "	690.5 "	307.6 "
$d_{.025}h$	36.0 "	640.5 "	284.2 "
$d_{.05}h$	34.5 "	553.5 "	260.7 "
$d_{.075}h$	31.5 "	530.5 "	246.8 "
$d_{.1}h$	30.5 "	517.0 "	237.2 "
$d_{.125}h$	30.0 "	511.0 "	230.8 "
$d_{.15}h$	30.5 "	500.5 "	225.1 "
$d_{.175}h$	28.5 "	488.5 "	220.1 "
$d_{.2}h$	27.0 "	478.5 "	216.0 "
$d_{.25}h$	26.0 "	471.5 "	208.0 "
$d_{.3}h$	25.5 "	448.0 "	200.7 "
$d_{.35}h$	25.0 "	430.0 "	193.8 "
$d_{.4}h$	24.0 "	419.5 "	186.6 "
$d_{.45}h$	23.5 "	411.0 "	178.6 "
$d_{.5}h$	22.5 "	394.0 "	169.4 "
$d_{.55}h$	20.5 "	380.5 "	159.8 "
$d_{.6}h$	20.0 "	371.0 "	149.4 "
$d_{.65}h$	17.5 "	336.5 "	137.1 "
$d_{.7}h$	16.5 "	309.5 "	123.3 "
$d_{.75}h$	14.5 "	259.5 "	108.1 "
$d_{.8}h$	13.0 "	213.5 "	91.0 "
$d_{.85}h$	10.0 "	187.5 "	71.5 "
$d_{.9}h$	9.0 "	155.5 "	49.9 "
$d_{.95}h$	5.0 "	115.5 "	28.3 "
h	2.75 m	32.70 m	16.77 m
$(h-h_c)/h$	0.032	0.906	0.482
age	15 years	242 years	104 years

point where the knot bump did not affect the diameter.

The minima, maxima, and arithmetic means of the sample tree measurements are given in Table 2. The diameters are expressed as the means of the cross measurements.

3. SIMULTANEOUS EQUATION MODELS

31. Regression equations

In our previous studies, the relative-height diameters were considered as endogenous variables and the height of the tree as an exogenous variable of the simultaneous equation model. In principle, however, all variables describing the tree can be seen as endogenous variables and those describing its enviroment as exogenous variables. The simultaneous equation model can be written as follows:

$$x_i = f(x_1, \ldots, x_j, \ldots, x_n) \quad i = 1, \ldots, m; \; i \neq j; \; m \leq n$$

where

x_k = variable k
m = nuber of endogeneous variables
n–m = number of exogenous variables

In this paper the regression equation which predicts the height of the tree is included in the simultaneous model. Consequently, the height is an endogenous variable of the simultaneous model. However, height as well as any relative-height diameter can be exogenized by giving it a measured value. It must be noticed that the derivation of the regression equations is not affected by the differentiation of the regressors into endogenous and exogenous variables.

One problem is the optimal number of the relative-height diameters as endogenous variables. The computational work clearly increases with increasing number of diameters. If we want to know, for example, only three diameters say $d_{.1}h$, $d_{.3}h$, and $d_{.6}h$, and have measured one of them, there is no value in including other diameters as endogenous variables in the simultaneous equation model. On the other hand, decreasing diameter intervals tends to decrease the error in the estimation of the intermediate values. Reliable intermediate values are important both in connecting the taper curve to the diameters measured at arbitrary heights, and in the final estimation of the taper curve for volume determination. Consequently, it was decided to employ all 24 relative-height diameters, as well as the height of the tree, as endogenous variables in the model. Nevertheless, the results would have been similar with only 13...15 diameters as endogenous variables.

The applicability of the tree and stand characteristics as exogenous variables was tested. The age of the tree and the crown ratio proved to be significant variables. The age was omitted from the final equations, mainly because it is difficult to measure. The numbers of the stands proved to be significant predicting regressors when used as dummy variables in the regression equations. This indicates that there may be stand characteristics which explain the variation of the taper curve. The number of stands, however, was too small to give any positive proof of these characteristics.

Logarithm transformations were made of the dependent variables in order to homogenize the residual variances. Both the first and second powers of the logarithmic values of the independent variables were tested. Due to the high correlation between the first and second powers, the computing accuracy was not always sufficient. Therefore, the diameters were replaced by diameter-height ratios (see KILKKI and VARMOLA 1979, p. 295) before the transformations. The second powers of the logarithmic diameter-height ratios did not markedly improve the equations.

Because the computing accuracy no longer required the employment of the diameter-height ratios, the logarithmic diameters instead of the logarithmic diameter-height ratios were used in the final models. Thus, the regression equations are logarithmically linear with respect to the diameters.

This simplification produced a major computational advantage, since standard matrix operations could be employed in solving the system of simultaneous equations whenever only diameters were endogenous variables.

The regression equations of the two simultaneous models and the residual standard errors of the equations are presented in Tables 3 and 4. In the calculation

Table 3. Regression coefficients and standard errors of the estimates of the regression equations for estimating the relative-height diameters and height when the crown ratio is unknown.
Taulukko 3. Suhteellisia osakorkeuksia edustavia läpimittoja ja puun pituutta ennustavien regressioyhtälöiden kertoimet ja keskivirheet, kun latvussuhdetta ei tunneta.

Independent variables *Selittävät muuttujat*	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
	Dependent variable – *Selitettävä muuttuja*							
	$\ln d_{.01}h$	$\ln d_{.025}h$	$\ln d_{.05}h$	$\ln d_{.075}h$	$\ln d_{.1}h$	$\ln d_{.125}h$	$\ln d_{.15}h$	$\ln d_{.175}h$
$\ln d_{.01}h$		.39366	–.03255	–.00627	–.02680	.04465	–.02991	.00962
$\ln d_{.025}h$	1.07963		.49349	.08453	.00414	–.06849	–.03803	–.02536
$\ln d_{.05}h$	–.11506	.63600		.31665	.14853	.04820	.09209	–.03348
$\ln d_{.075}h$	–.02587	.12709	.36941		.39374	.19008	.03773	.03426
$\ln d_{.1}h$	–.10914	.00615	.17117	.38894		.25259	–.02413	.15365
$\ln d_{.125}h$	.17699	–.09899	.05406	.18273	.24583		.34068	.08421
$\ln d_{.15}h$	–.13493	–.06254	.11752	.04127	–.02672	.38764		.33606
$\ln d_{.175}h$	.04300	–.04132	–.04233	.03712	.16856	.09493	.33293	
$\ln d_{.2}h$	.16436	–.03052	–.04463	–.05847	.14151	–.02003	.27258	.20269
$\ln d_{.25}h$	–.07219	.05677	–.12595	.08893	–.04386	.06657	.00768	.17272
$\ln d_{.3}h$	–.11232	.05995	.01563	–.00264	–.06635	.06038	–.06057	.10885
$\ln d_{.35}h$	–.13315	.06171	–.00474	–.06511	.01498	.03262	.05397	–.06378
$\ln d_{.4}h$	.17793	–.03627	–.00183	–.05063	.08754	–.04453	–.00664	.03126
$\ln d_{.45}h$	.05183	–.04042	.01940	–.00319	–.01995	–.02482	.02848	.04093
$\ln d_{.5}h$	.06298	–.04570	–.01839	.05690	–.03283	.01328	.00956	–.05463
$\ln d_{.55}h$	–.10233	.00163	.05931	–.01865	–.01357	.01741	–.02005	–.01858
$\ln d_{.6}h$	–.00224	.01498	–.01188	.03286	–.00175	–.05042	–.00691	.02838
$\ln d_{.65}h$	–.00021	.02560	–.03259	–.02319	.04273	.00588	.00279	–.02840
$\ln d_{.7}h$	.04434	–.02502	.00371	.01432	–.00617	–.01773	.01356	.03271
$\ln d_{.75}h$	.00938	.01327	.01779	–.04299	–.00124	–.00158	–.00646	.00891
$\ln d_{.8}h$	–.04071	.00275	.00428	.02622	–.02825	.02699	–.02951	–.00669
$\ln d_{.85}h$	–.00731	.01172	–.00936	.00028	.01471	–.01740	.02665	–.00723
$\ln d_{.9}h$	.00229	–.01341	.00490	.00362	–.00693	.01029	.00132	–.01365
$\ln d_{.95}h$	.01103	–.00122	–.00411	.00073	.00329	–.00760	–.00347	.01166
$\ln h$	.06594	–.08580	.06389	.01044	.01335	–.02898	–.02126	.01924
$(\ln h)^2$	–.00599	.01574	–.01460	–.00358	–.00079	.00478	.00515	–.00367
constant	.12808	.02971	–.08209	–.01352	.01485	.02691	.06489	–.04317
standard error	.03469	.02094	.01844	.01708	.01719	.01742	.01633	.01641

of the residual standard errors, the loss of the degrees of freedom was not taken into account (cf. KILKKI and VARMOLA 1979, p. 299).

In model I (Table 3), there are 25 equations without the crown ratio and in model II (Table 4) 25 equations with the crown ratio as an independent variable. The equations with the stand numbers as dummy variables are not given since they are not of general interest.

Model II, plus one equation in which the crown ratio is the dependent variable, would have comprised all the information given by model I. However, the crown ratio equation would have been nonlinear with respect to the logarithmic diameters. Therefore, the logarithmically linear system of equations without crown ratio (model I) is also presented.

The residual variances of the regression equations were not homogeneous with respect to the size of the tree. Therefore, slight biases were discernible in the models with respect to the size of the tree. Due to the inadequate data no weighted regression analysis was applied.

Regression models were derived to predict

Table 3. Continued.
Taulukko 3. Jatkoa.

Independent variables *Selittävät muuttujat*	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)
	Dependent variable – *Selitettävä muuttuja*							
	$\ln d_{.2}h$	$\ln d_{.25}h$	$\ln d_{.3}h$	$\ln d_{.35}h$	$\ln d_{.4}h$	$\ln d_{.45}h$	$\ln d_{.5}h$	$\ln d_{.55}h$
$\ln d_{.01}h$	.04531	–.02139	–.03717	–.05112	.06436	.02308	.03862	–.05819
$\ln d_{.025}h$	–.02308	.04612	.05441	.06498	–.03598	–.04936	–.07686	.00254
$\ln d_{.05}h$	–.04349	–.13189	.01828	–.00643	–.00234	.03054	–.03985	.11922
$\ln d_{.075}h$	–.06646	.10864	–.00360	–.10307	–.07552	–.00585	.14389	–.04374
$\ln d_{.1}h$	.15890	–.05292	–.08945	.02343	.12896	–.03618	–.08200	–.03143
$\ln d_{.125}h$	–.02189	.07817	.07921	.04964	–.06384	–.04381	.03228	.03924
$\ln d_{.15}h$	.33893	.01026	–.09042	.09346	–.01083	.05720	.02644	–.05143
$\ln d_{.175}h$	.24968	.22864	.16097	–.10942	.05052	.08142	–.14970	–.04721
$\ln d_{.2}h$		.25720	.08859	.16514	–.06193	–.06336	.02942	.08359
$\ln d_{.25}h$	.23933		.30860	.12572	.10717	.04812	.00620	.02051
$\ln d_{.3}h$	.07379	.27624		.28949	.20955	.03957	.07307	.01432
$\ln d_{.35}h$	.11858	.09701	.24955		.24958	.13163	.12686	–.08554
$\ln d_{.4}h$	–.04720	.08778	.19174	.26492		.39857	.14062	.10771
$\ln d_{.45}h$	–.03923	.03202	.02941	.11350	.32378		.22900	.24094
$\ln d_{.5}h$	.01323	.00300	.03943	.07942	.08294	.16627		.32278
$\ln d_{.55}h$	.04052	.01068	.00833	–.05775	.06850	.18863	.34806	
$\ln d_{.6}h$	–.03292	–.01105	–.00518	.08663	–.02721	.04438	.08143	.21285
$\ln d_{.65}h$	.01440	–.01168	–.00434	.00916	–.00673	–.01879	.09198	.09572
$\ln d_{.7}h$	–.03815	–.02395	–.01023	.00720	–.00895	.04050	.00356	.01504
$\ln d_{.75}h$	.01374	.00235	.02056	–.03683	.02516	–.02555	–.00275	.02334
$\ln d_{.8}h$	.02086	.02394	–.02302	–.01113	–.01831	.01929	–.04084	.02788
$\ln d_{.85}h$	–.01207	–.02964	.01569	.01241	–.00151	–.01664	.02926	–.01205
$\ln d_{.9}h$	.00321	.02795	–.02802	–.01586	.00992	.00447	–.00482	.00792
$\ln d_{.95}h$	–.00113	–.00898	.02013	.00325	–.00335	–.00911	–.00104	–.01098
$\ln h$	.00273	–.00466	.00181	–.02674	.00790	–.03335	.03293	–.01220
$(\ln h)^2$	–.00177	.00174	.00131	.00588	–.00243	.00500	–.00563	.00354
constant	–.02656	.01260	.03186	.03485	–.04229	.01211	–.05485	.02839
standard error	.01821	.01888	.01995	.02149	.02086	.02314	.02716	.02615

the residual deviation of the relative-height diameter as a function of the height of the tree. These models had the following form:

$$y = a + bh \qquad (51)$$

where

$y = \sqrt{\left| \ln d_{xh} - \widehat{\ln d_{xh}} \right|}$

h = height of the tree

a, b = regression coefficients

$\ln d_{xh}$ = logarithm of the measured diameter at the relative height x

$\widehat{\ln d_{xh}}$ = estimated logarithmic diameter at the relative height x

The regression coefficients and the standard errors of the regression equations corresponding to the y values derived from equations (1) . . . (24) and (26) . . . (49) are given in Tables 5 and 6, respectively.

To derive unbiased residual variances from equation (51), the following formula, based upon Taylor's expansion, is needed:

Table 3. Continued.
Taulukko 3. Jatkoa.

Independent variables *Selittävät muuttujat*	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)	(25)
	Dependent variable – *Selitettävä muuttuja*								
	$\ln d_{.6}h$	$\ln d_{.65}h$	$\ln d_{.7}h$	$\ln d_{.75}h$	$\ln d_{.8}h$	$\ln d_{.85}h$	$\ln d_{.9}h$	$\ln d_{.95}h$	$\ln h$
$\ln d_{.01}h$	–.00217	–.00022	.06691	.01860	–.11065	–.03311	.01840	.22499	1.46660
$\ln d_{.025}h$	.03982	.07377	–.10358	.07221	.02053	.14561	–.29556	–.06802	–.94885
$\ln d_{.05}h$	–.04069	–.12103	.01978	.12475	.04116	–.14999	.13921	–.29650	–1.09389
$\ln d_{.075}h$	.13133	–.10048	.08914	–.35170	.29386	.00532	.11995	.06107	–1.17744
$\ln d_{.1}h$	–.00692	.18286	–.03792	–.01001	–.31270	.27145	–.22694	.27294	1.54858
$\ln d_{.125}h$	–.19371	.02450	–.10609	–.01244	.29083	–.31254	.32782	–.61447	–.88051
$\ln d_{.15}h$	–.03020	.01321	.09229	–.05782	–.36180	.54480	.04792	–.31924	.72318
$\ln d_{.175}h$	.12291	–.13333	.22057	.07899	–.08122	–.14642	–.48995	1.06249	.22102
$\ln d_{.2}h$	–.11572	.05489	–.20884	.09892	.20571	–.19846	.09349	–.08384	–.86441
$\ln d_{.25}h$	–.03613	–.04143	–.12203	.01574	.21966	–.45337	.75788	–.61802	.52419
$\ln d_{.3}h$	–.01516	–.01378	–.04666	.12329	–.18905	.21489	–.68018	1.24037	.99994
$\ln d_{.35}h$	.21870	.02507	.02832	–.19034	–.07877	.14645	–.33194	.17256	.21904
$\ln d_{.4}h$	–.07292	–.01954	–.03734	.13805	–.13757	–.01889	.22032	–.18914	–.44241
$\ln d_{.45}h$	.09662	–.04435	.13728	–.11385	.11775	–.16939	.08064	–.41724	–.79617
$\ln d_{.5}h$	.12870	.15760	.00876	–.00890	–.18100	.21626	–.06310	–.03459	.34747
$\ln d_{.55}h$	.36274	.17686	.03992	.08143	.13323	–.09604	.11194	–.39369	.36376
$\ln d_{.6}h$		.34536	.11722	.04399	–.03907	.00789	.12754	–.13697	.18145
$\ln d_{.65}h$	.31856		.45042	.20902	.06791	–.15951	.09047	.02683	–.32026
$\ln d_{.7}h$	.07527	.31358		.25816	.31545	.09441	–.08025	01987	.25260
$\ln d_{.75}h$	.02149	.11068	.19636		.49117	.31312	–.00531	–.22229	.22224
$\ln d_{.8}h$	–.01393	.02626	.17516	.35857		.41898	.03549	.18154	.30390
$\ln d_{.85}h$	.00169	–.03699	.03144	.13710	.25129		.60785	.15628	–.12858
$\ln d_{.9}h$	.01539	.01183	–.01508	–.00131	.01201	.34287		.92932	.22158
$\ln d_{.95}h$	–.00651	.00138	.00147	–.02162	.02419	.03472	.36603		–.30651
$\ln h$	.05206	.03105	–.00362	.06488	–.03821	.02624	–.13625	–.42010	
$(\ln h)^2$	–.00972	–.00804	.00268	–.01099	.01199	–.00831	.03683	.05390	
constant	–.06934	–.05086	.01374	–.05490	.05779	–.10012	.30463	.30949	–.48826
standard error	.03414	.03555	.04261	.04886	.05718	.07383	.09830	.15663	.21946

$$\hat{s}^2_{\ln d_x h} = \hat{y}^4 + 6\hat{y}^2 s_{\hat{y}}^2 + 3 s_{\hat{y}}^4 \qquad (52)$$

where

$\hat{s}^2_{\ln d_x h}$ = expected value of the residual variance of the logarithmic relative-height diameter equation (equations (1) . . . (24) and (26) . . . (49))

$\hat{y}$ = estimated value from equation (51)

$s_{\hat{y}}$ = residual standard deviation of the equation (51)

32. Combination of the measured tree characteristics with the simultaneous model

Any of the variables of the regression equations (1) . . . (25) and (26) . . . (50) may assume measured values. Then, the regression equation with this variable as a dependent variable can be omitted. In the other equations, the variable assumes the measured value and may be treated as an exogenous variable. Thus, the number of the endogenous variables is reduced by one. Another possibility is to replace the regression equation by a new equation which states the equivalence between the dependent variable and its measured value. Then, the rank of the original system remains unchanged.

If the height of a diameter measurement does not coincide with one of the prefixed relative heights, interpolation has to be applied. The interpolation formula is incorporated into the simultaneous equation model as a new equation. Since the number of the endogenous variables and the number of the equations have to be equal, one of the regression equations has to be dropped. Usually, the best choice is the regression equation representing the diameter closest to the measured diameter.

Table 4. Regression coefficients and standard errors of the estimates of the regression equations for estimating the relative-height diameters and height when the crown ratio is known.
Taulukko 4. Suhteellisia osakorkeuksia edustavia läpimittoja ja puun pituutta ennustavien regressioyhtälöiden kertoimet ja keskivirheet, kun latvussuhde tunnetaan.

Independent variables *Selittävät muuttujat*	(26)	(27)	(28)	(29)	(30)	(31)	(32)	(33)
	Dependent variable – *Selitettävä muuttuja*							
	$\ln d_{.01}h$	$\ln d_{.025}h$	$\ln d_{.05}h$	$\ln d_{.075}h$	$\ln d_{.1}h$	$\ln d_{.125}h$	$\ln d_{.15}h$	$\ln d_{.175}h$
$\ln d_{.01}h$		.39447	–.03411	–.00418	–.01888	.04620	–.03380	.00968
$\ln d_{.025}h$	1.07333		.49641	.08064	.00636	–.07068	–.02770	–.03120
$\ln d_{.05}h$	–.11856	.63413		.31771	.14342	.04917	.08623	–.03029
$\ln d_{.075}h$	–.01696	.12020	.37074		.38436	.18869	.04228	.03108
$\ln d_{.1}h$	–.07741	.00957	.16915	.38846		.25164	–.02773	.16211
$\ln d_{.125}h$	.17991	–.10116	.05509	.18117	.23907		.33852	.08212
$\ln d_{.15}h$	–.15194	–.04575	.11151	.04686	–.03040	.39072		.34277
$\ln d_{.175}h$	.04281	–.05073	–.03855	.03390	.17494	.09328	.33732	
$\ln d_{.2}h$	.15825	–.03637	–.04219	–.06023	.14704	–.02085	.27444	.19496
$\ln d_{.25}h$	–.09185	.06373	–.12829	.09173	–.03657	.06862	.00080	.17306
$\ln d_{.3}h$	–.11523	.06023	.01533	–.00221	–.06228	.06069	–.06043	.10760
$\ln d_{.35}h$	–.13761	.06353	–.00556	–.06401	.01668	.03328	.05096	–.06268
$\ln d_{.4}h$	.15994	–.03282	–.00278	–.04915	.09224	–.04336	–.00915	.03031
$\ln d_{.45}h$	.05511	–.04590	.02154	–.00518	–.01585	–.02591	.03359	.03644
$\ln d_{.5}h$	.05374	–.04817	–.01713	.05607	–.02312	.01316	.01240	–.05856
$\ln d_{.55}h$	–.10270	.00207	.05907	–.01841	–.01218	.01756	–.02014	–.01853
$\ln d_{.6}h$	.01135	.01419	–.01181	.03122	–.01044	–.05108	–.00649	.03066
$\ln d_{.65}h$	.01242	.02396	–.03219	–.02393	.03441	.00511	.00378	–.02648
$\ln d_{.7}h$	.05885	–.02629	.00407	.01339	–.01499	–.01863	.01449	.03458
$\ln d_{.75}h$	.00381	.01385	.01756	–.04252	.00174	–.00123	–.00694	.00824
$\ln d_{.8}h$	–.03552	.00446	.00356	.02660	–.03258	.02705	–.03082	–.00412
$\ln d_{.85}h$	–.00571	.01233	–.00965	.00048	.01250	–.01733	.02552	–.00620
$\ln d_{.9}h$	.00479	–.01306	.00478	.00361	–.00877	.01021	.00105	–.01277
$\ln d_{.95}h$	.00499	.00109	–.00489	.00161	.00473	–.00698	–.00548	.01222
$\ln h$	.06499	–.08289	.06302	.01106	.01061	–.02866	–.02286	.02088
$(\ln h)^2$	–.00513	.01480	–.01430	–.00384	–.00044	.00462	.00579	–.00414
$\ln(h)_c/h)$	.04311	–.01005	.00319	–.00438	–.01885	–.00347	.00867	.00107
$(\ln(h)_c/h))^2$	.01484	–.00601	.00206	–.00232	–.00365	–.00160	.00542	–.00184
constant	.16588	.00238	–.07986	–.01682	–.00568	.02403	.06976	–.03978
standard error	.03432	.02083	.01843	.01706	.01698	.01741	.01621	.01633

The interpolation formulas given in our previous papers (KILKKI et al. 1978; KILKKI and VARMOLA 1979) together with the Hermitian interpolation and natural cubic spline interpolation (see e.g. KIESEWETTER and MAESS 1974; LAHTINEN and LAASASEN-AHO 1979; IMSL Library 2, 1977; FRÖBERG 1972) were tested.

Differences in the results when different formulas were applied were insignificant. Even linear interpolation yielded acceptable results for a major part of the stem. This results is due to the small intervals between the relative-height diameters. In the following, the natural cubic spline interpolation is employed. In order to make the estimation more reliable, two imaginary diameter values were fixed both below the stump and above the top of the stem:

Table 4. Continued.
Taulukko 4. Jatkoa.

Independent variables *Selittävät muuttujat*	(34)	(35)	(36)	(37)	(38)	(39)	(40)	(41)
	Dependet variable – *Selitettävä muuttuja*							
	$\ln d_{.2}h$	$\ln d_{.25}h$	$\ln d_{.3}h$	$\ln d_{.35}h$	$\ln d_{.4}h$	$\ln d_{.45}h$	$\ln d_{.5}h$	$\ln d_{.55}h$
$\ln d_{.01}h$	.04419	–.02751	–.03883	–.05376	.05866	.02488	.03328	–.05948
$\ln d_{.025}h$	–.02763	.05193	.05522	.06753	–.03275	–.05639	–.08117	.00326
$\ln d_{.05}h$	–.04095	–.13355	.01795	–.00755	–.00354	.03380	–.03688	.11892
$\ln d_{.075}h$	–.06822	.11143	–.00302	–.10142	–.07312	–.00948	.14084	–.04324
$\ln d_{.1}h$	.16832	–.04489	–.08603	.02672	.13869	–.02934	–.05869	–.02892
$\ln d_{.125}h$	–.02267	.08004	.07964	.05064	–.06194	–.04557	.03174	.03962
$\ln d_{.15}h$	.34451	.00107	–.09153	.08949	–.01509	.06818	.03451	–.05244
$\ln d_{.175}h$	.24085	.22927	.16039	–.10832	.04918	.07278	–.16040	–.04749
$\ln d_{.2}h$		.25574	.08786	.16515	–.06350	–.06870	.01953	.08312
$\ln d_{.25}h$	.23847		.30665	.12232	.10075	.05169	.00279	.01908
$\ln d_{.3}h$	.07281	.27253		.28849	.20709	.03954	.07053	.01395
$\ln d_{.35}h$	.11805	.09377	.24884		.24651	.13226	.12462	–.08600
$\ln d_{.4}h$	–.04835	.08226	.19025	.26255		.39754	.13423	.10661
$\ln d_{.45}h$	–.04249	.03429	.02951	.11444	.32295		.22220	.24104
$\ln d_{.5}h$	.00881	.00135	.03838	.07862	.07950	.16201		.32198
$\ln d_{.55}h$	.04007	.00986	.00811	–.05800	.06751	.18789	.34424	
$\ln d_{.6}h$	–.02975	–.00699	–.00376	.08825	–.02269	.04534	.08823	.21388
$\ln d_{.65}h$	.01650	–.00772	–.00307	.01082	–.00270	–.01829	.09720	.09667
$\ln d_{.7}h$	–.03519	–.01910	–.00868	.00921	–.00404	.04094	.01100	.01622
$\ln d_{.75}h$	.01284	.00065	.02002	–.03751	.02336	–.02549	–.00512	.02293
$\ln d_{.8}h$	.02323	.02463	–.02243	–.01073	–.01652	.02123	–.03557	.02829
$\ln d_{.85}h$	–.01106	–.02917	.01589	.01250	–.00092	–.01576	.03084	–.01191
$\ln d_{.9}h$	.00401	.02841	–.02772	–.01555	.01074	.00495	–.00295	.00813
$\ln d_{.95}h$	–.00085	–.01107	.01962	.00232	–.00497	–.00786	–.00174	–.01136
ln h	.00434	–.00500	.00195	–.02687	.00822	–.03146	.03533	–.01211
$(\ln h)^2$	–.00216	.00211	.00135	.00604	–.00227	.00442	–.00609	.00357
$\ln(h_c/h)$	.00328	.01443	.00402	.00619	.01293	–.00313	.01380	.00311
$(\ln(h_c/h))^2$	–.00084	.00554	.00124	.00237	.00411	–.00324	.00154	.00098
constant	–.02154	.02541	.03572	.04037	–.02974	.01114	–.03857	.03136
standard error	.01816	.01880	.01995	.02148	.02081	.02309	.02704	.02615

$$d_{-.005}h = 2.1\ d_{.01}h - d_{.025}h \qquad (53)$$

$$d_{1.05}h = 2\ d_h - d_{.95}h \qquad (54)$$

The top diameter (d_h) always assumes a constant value of 5 millimeters.

The regression equations (1) . . . (25) and (26) . . . (50) are nonlinear. To solve the simultaneous models I and II directly would have consumed a great deal of computer time (see KILKKI and VARMOLA 1979, p. 300). Consequently, another solution algorithm was employed. The algorithm is based upon the assumption that the height of the tree is known and thus can be considered as an exogenous variable. Then, the height equation may be dropped from the system. The simultaneous model is now linear with respect to the logarithmic endogenous variables, i.e. the logarithmic diameters.

If the diameters have been measured at some of the prefixed relative heights, the respective logarithmic equations of the system are replaced by new equations:

$$\ln d_{xh} = \ln dl_{xh} \qquad (55)$$

where

dl_{xh} = measured value of the diameter at the relative height x

Table 4. Continued.
Taulukko 4. Jatkoa.

Independent variables *Selitettävät muuttujat*	(42)	(43	(44)	(45)	(46)	(47)	(48)	(49)	(50)
	Dependent variable – *Selitettävä muuttuja*								
	$\ln d_{.6}h$	$\ln d_{.65}h$	$\ln d_{.7}h$	$\ln d_{.75}h$	$\ln d_{.8}h$	$\ln d_{.85}h$	$\ln d_{.9}h$	$\ln d_{.95}h$	ln h
$\ln d_{.01}h$	.01105	.01315	.08837	.00768	–.09713	–.02628	.03901	.09937	1.56509
$\ln d_{.025}h$	.03761	.06902	–.10743	.07591	.03322	.15447	–.28925	.05892	–1.13108
$\ln d_{.05}h$	–.03998	–.11845	.02122	.12291	.03384	–.15434	.13519	–.33846	–.92687
$\ln d_{.075}h$	.12732	–.10274	.08153	–.34734	.29503	.00896	.11921	.13040	–1.22452
$\ln d_{.1}h$	–.04167	.14932	–.09226	.01437	–.36523	.23594	–.29266	.38621	1.36365
$\ln d_{.125}h$	–.19377	.02106	–.10896	–.00965	.28808	–.31070	.32353	–.54142	–.92758
$\ln d_{.15}h$	–.02841	.01801	.09781	–.06282	–.37889	.52798	.03830	–.49058	.95766
$\ln d_{.175}h$	.13213	–.12402	.22969	.07340	–.04984	–.12622	–.45994	1.07673	.11416
$\ln d_{.2}h$	–.10379	.06254	–.18923	.09263	.22749	–.18229	.11674	–.06096	–.87442
$\ln d_{.25}h$	–.02274	–.02728	–.09575	.00435	.22489	–.44826	.77199	–.73648	.69499
$\ln d_{.3}h$	–.01088	–.00964	–.03867	.11965	–.18205	.21699	–.66959	1.16035	.99302
$\ln d_{.35}h$	.22006	.02932	.03540	–.19339	–.07511	.14733	–.32405	.11832	.26285
$\ln d_{.4}h$	–.06027	–.00780	–.01654	.12829	–.12318	–.01153	.23842	–.26985	–.30556
$\ln d_{.45}h$	.09783	–.04288	.13614	–.11371	.12861	–.16063	.08918	–.34666	–.85947
$\ln d_{.5}h$	.13880	.16616	.02668	–.01666	–.15709	.22921	–.03876	–.05584	.35228
$\ln d_{.55}h$	.35972	.17669	.04204	.07972	.13356	–.09463	.11429	–.39095	.36140
$\ln d_{.6}h$		.32911	.09278	.05363	–.05721	–.00336	.10277	–.06223	.10094
$\ln d_{.65}h$	.30286		.42090	.21693	.05220	–.16791	.06979	.09449	–.38508
$\ln d_{.7}h$	.06019	.29674		.26783	.29309	.08311	–.10470	.10182	.14070
$\ln d_{.75}h$	.02594	.11403	.19969		.49104	.31572	.00294	–.24329	.25462
$\ln d_{.8}h$	–.02038	.02021	.16093	.36163		.41006	.02175	.18591	.28280
$\ln d_{.85}h$	–.00071	–.03862	.02711	.13814	.24362		.59958	.15280	–.12185
$\ln d_{.9}h$	.01230	.00908	–.01932	.00073	.00731	.33909		.90229	.19748
$\ln d_{.95}h$	–.00304	.00502	.00767	–.02459	.02552	.03530	.36857		–.23926
ln h	.04876	.02874	–.00708	.06602	–.04452	.02178	–.14181	–.41283	
$(\ln h)^2$	–.00947	–.00801	.00255	–.01084	.01315	–.00739	.03743	.05963	
$\ln(h)_c/h)$	–.03292	–.03281	–.05665	.02648	–.03373	–.01915	–.05339	.26486	–.34436
$(\ln(h_c/h))^2$	–.00836	–.00911	–.01554	.00762	–.00350	–.00132	–.01099	.10544	–.14631
Constant	–.10181	–.08287	–.04298	–.02867	.01819	–.12252	.24644	.53033	–.68967
standard error	.03391	.03536	.04211	.04876	.05682	.07372	.09804	.15338	.21493

This new set of equations is solved by standard matrix operations. The solution gives unbiased estimates for the logarithmic diameters. In order to obtain unbiased diameter estimates, the logarithmic diameter estimates have to be corrected before taking antilogarithms (see e.g. MEYER 1941). This correction is executed by increasing each logarithmic diameter estimate by one half of its error variance:

$$d_{xh} = \exp(\widehat{\ln d_{xh}} + \frac{v_i}{2}) \qquad (56)$$

Formula (58) which gives the value for v_i is derived as follows. If we denote the difference between the measured and estimated variable x_i by $\triangle_i$, we can derive the following formula for the covariance of the errors:

$$\begin{aligned} \mathrm{cov}(\triangle_i\triangle_j) &= E((x_i - E(x_i))(x_j - E(x_j)) \\ &= E(x_ix_j - x_iE(x_j) - E(x_i)x_j + E(x_i)E(x_j)) \\ &= E(x_ix_j) - E(x_i)E(x_j) - E(x_i)E(x_j) + E(x_i)E(x_j) \\ &= E(x_ix_j) - E(x_i)E(x_j) \end{aligned}$$

Table 5. Regression coefficients (a and b) and standard errors of the estimates (s) for formula (51) when the crown ratio is unknown.
Taulukko 5. Kaavan (51) regressiokertoimet (a ja b) ja keskivirheet (s), kun latvussuhdetta ei tunneta.

Diameter *Läpimitta*	a	b	s
$d_{.01h}$	.17282	−.00156	.06780
$d_{.025h}$	.13952	−.00135	.05003
$d_{.05h}$	.13178	−.00141	.04909
$d_{.075h}$	.12697	−.00149	.04540
$d_{.1h}$	.14527	−.00260	.04403
$d_{.125h}$	.13281	−.00195	.04761
$d_{.15h}$	.12829	−.00175	.04617
$d_{.175h}$	.12089	−.00122	.04601
$d_{.2h}$	.13115	−.00165	.04943
$d_{.25h}$	.12694	−.00126	.04911
$d_{.3h}$	.13165	−.00136	.05156
$d_{.35h}$	.14792	−.00203	.05138
$d_{.4h}$	.13926	−.00155	.05239
$d_{.45h}$	.14138	−.00130	.05407
$d_{.5h}$	.15969	−.00209	.06079
$d_{.55h}$	.16212	−.00216	.05715
$d_{.6h}$	.17713	−.00217	.06703
$d_{.65h}$	.17183	−.00156	.06920
$d_{.7h}$	.19820	−.00218	.07488
$d_{.75h}$	.21204	−.00234	.07989
$d_{.8h}$	.21832	−.00172	.08528
$d_{.85h}$	.24346	−.00194	.10205
$d_{.9h}$	.27041	−.00144	.11376
$d_{.95h}$	.34464	−.00172	.14359

By denoting:

b_{ij} = ij[th] element of the inverse matrix of the coefficient matrix of the endogenous variables in the simultaneous equation models I and II

c_k = the constant (including the effect of the exogenous variables) of the k[th] equation

ε_k = error in the k[th] equation

m = number of endogenous variables

we can develop the formula further:

$$\mathrm{cov}(\triangle_i\triangle_j) = E\left(\left(\sum_{k=1}^{m} b_{ik}c_k + \sum_{k=1}^{m} b_{ik}\varepsilon_k\right)\left(\sum_{l=1}^{m} b_{jl}c_l + \sum_{l=1}^{m} b_{jl}\varepsilon_l\right)\right)$$

$$- E\left(\sum_{k=1}^{m} b_{ik}c_k\right)E\left(\sum_{l=1}^{m} b_{jl}c_l\right)$$

$$= E\left(\sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}c_kc_l + \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}c_k\varepsilon_l + \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}\varepsilon_kc_l + \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}\varepsilon_k\varepsilon_l\right)$$

$$- \left(\sum_{k=1}^{m} b_{ik}c_k\right)\left(\sum_{l=1}^{m} b_{jl}c_l\right)$$

$$= \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}c_kc_l + \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}\mathrm{cov}_{kl}$$

$$- \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}c_kc_l$$

$$= \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}\mathrm{cov}_{kl} = \sum_{k=1}^{m}\sum_{l=1}^{m} b_{ik}b_{jl}s_ks_lr_{kl} \qquad (57)$$

where cov_{kl} = covariance between the residuals of the k[th] and l[th] regressions equations

s_k = standard error of the k[th] regression equation (either from Tables 3 and 4 or from formula 52)

r_{kl} = correlation coefficient between the residuals of the k[th] and l[th] regression equations (Tables 7 and 8)

If the endogenous variables assume measured values, the standard errors (s_k) of the respective equations equal zero.

The error variance of the endogenous variable x_i is:

$$v_i = \mathrm{cov}(\triangle_i\triangle_i) \qquad (58)$$

If the diameters have been measured at some other heights than the prefixed relative heights, the system of equations should comprise as many interpolation formulas as there are measured diameters, given that the measurement heights are not too close to each other. However, if the Hermitian or cubic spline interpolation formulas, for example, are employed, they cannot be linearized by taking logarithms. Therefore an iterative algorithm has to be applied (Fig. 2).

First, the Hermitian interpolation formula is written to estimate each measured diameter from the four relative-height diameters, two on both sides of the measured diameter. The coefficients of the closest relative-height diameter equation (Tables 3 and 4) are replaced by the partial derivatives of this interpolation formula with respect to the relative-height diameters. The partial derivatives A1, A2, A3, and A4 (= the nonzero coefficients) of the Hermitian interpolation formula with respect to the four closest relative-height diameters of the measured diameter are as follows (see KIESEWETTER and MAESS 1974, p. 148):

Table 6. Regression coefficients (a and b) and standard errors of estimates (s) for formula (51) when the crown ratio is known.
Taulukko 6. Kaavan (51) regressiokertoimet (a ja b) ja keskivirheet (s), kun latvussuhde tunnetaan.

Diameter *Läpimitta*	a	b	s
$d_{.01}h$	.17360	–.00164	.06742
$d_{.025}h$	.13751	–.00125	.05018
$d_{.05}h$	.13160	–.00140	.04886
$d_{.075}h$	.12664	–.00146	.04524
$d_{.1}h$	.14332	–.00253	.04475
$d_{.125}h$	.13238	–.00194	.04773
$d_{.15}h$	.12898	–.00181	.04575
$d_{.175}h$	.12098	–.00125	.04592
$d_{.2}h$	.13300	–.00176	.04926
$d_{.25}h$	.12439	–.00113	.04951
$d_{.3}h$	.13123	–.00136	.05206
$d_{.35}h$	.14846	–.00207	.05134
$d_{.4}h$	.13805	–.00151	.05288
$d_{.45}h$	.14154	–.00131	.05389
$d_{.5}h$	.16011	–.00214	.06089
$d_{.55}h$	.16201	–.00215	.05712
$d_{.6}h$	.17836	–.00230	.06717
$d_{.65}h$	.17325	–.00160	.06820
$d_{.7}h$	.19468	–.00197	.07339
$d_{.75}h$	.21417	–.00242	.07854
$d_{.8}h$	.21709	–.00172	.08607
$d_{.85}h$	.24359	–.00195	.10182
$d_{.9}h$	.27051	–.00148	.11415
$d_{.95}h$	.34451	–.00194	.14279

A1 = (H2–H3)/((H1–H2)*(H1–H3))*F3
A2 = F1+F3*(2*H2–H3–H1)/((H2–H1)*(H2–H3))+F4*(H3–H4)/((H2–H3)*(H2–H4))
A3 = F2+F3*(H2–H1)/((H3–H1)*(H3–H2))+F4*(2*H3–H4–H2)/((H3–H2)*(H3–H4))
A4 = F4*(H3–H2)/((H4–H2)*(H4–H3))

where

F1 = 1–3*S**2+2*S**3
F2 = 3*S**2–2*S**3
F3 = (H3–H2)*(S–2*S**2+S**3)
F4 = (H3–H2)*(S**3–S**2)
S = (HR–H2)/(H3–H2)
HR = relative height of the measured diameter (H2 ≦ HR ≦ H3)
H1, H2, H3, and H4 = four successive prefixed relative heights closest to HR

The inverse of the new matrix is now employed in formula (57). The standard errors of the two equations on both sides of each measured diameter are estimated by the following heuristic formula:

$$s_k' = s_k \sqrt{|f|} \quad (59)$$

where

s_k' = new standard error of equation k
s_k = original standard error of equation k
f = relative distance of the measured diameter from diamerer k. The distance between diameters k and k+1 (or k–1) equals 1.

Table 7. Correlation coefficients between the residuals of the regression equations (1) . . . (24).
Taulukko 7. Regressioyhtälöiden (1) . . . (24) residuaalien väliset korrelaatiokertoimet.

	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)	(13)	(14)	(15)	(16)	(17)	(18)	(19)	(20)	(21)	(22)	(23)	(24)
(1)	–.651	.064	.008	.058	–.091	.070	–.026	–.091	.044	.068	.080	–.114	–.031	–.047	.075	.003	–.001	–.053	–.013	.067	.014	–.006	–.049
(2)		–.563	–.099	–.009	.086	.043	.041	028	–.058	–.062	–.060	.045	.038	.060	.000	–.027	–.042	.049	–.031	–.007	–.041	.063	.008
(3)			–.342	–.155	–.055	–.099	.028	.046	.135	–.015	.005	–.004	–.021	.025	–.082	.023	.061	–.008	–.048	–.012	.038	–.028	.037
(4)				–.394	–.185	–.048	–.029	.065	–.104	.007	.083	.063	.003	–.088	.025	–.066	.049	–.034	.122	–.087	–.003	–.017	–.009
(5)					–.249	.032	–.165	–.149	.052	.075	–.025	–.107	.035	.048	.018	.006	–.089	.016	.005	.092	–.062	.038	–.029
(6)						–.362	–.083	.018	–.071	–.078	–.036	.056	.029	–.019	–.022	.099	–.013	.041	.006	–.089	.075	–.060	.069
(7)							–.337	–.313	–.004	.078	–.068	.004	–.039	–.018	.030	.012	–.001	–.037	.018	.106	–.121	–.007	.032
(8)								–.220	–.206	–.131	.084	–.038	–.061	.094	.032	–.060	.057	–.083	–.025	.023	.031	.082	–.111
(9)									–.249	–.079	–.141	.056	.049	–.017	–.061	.062	–.028	.090	–.038	–.065	.049	–.017	.009
(10)										–.292	–.109	–.098	–.039	–.008	–.013	.020	.023	.054	–.007	–.073	.117	–.147	.076
(11)											–.271	–.202	–.034	–.053	–.011	.010	.008	.020	–.050	.067	–.059	.139	–.159
(12)												–.258	–.123	–.097	.070	–.139	–.013	–.016	.086	.029	–.043	.072	–.023
(13)													–.356	–.108	–.087	.046	.008	.021	–.060	.050	.005	–.046	.025
(14)														–.198	–.212	–.067	.030	–.075	.053	–.047	.054	–.020	.062
(15)															–.334	–.104	–.120	–.005	.005	.086	–.080	.019	.005
(16)																–.277	–.131	–.024	–.044	–.061	.035	–.029	.066
(17)																	–.331	–.096	–.030	.023	–.003	–.045	.030
(18)																		–.375	–.152	–.042	.076	–.033	–.006
(19)																			–.225	–.235	–.054	.035	–.006
(20)																				–.420	–.208	.003	.069
(21)																					–.325	–.021	–.066
(22)																						–.457	–.073
(23)																							–.583

Table 8. Correlation coefficients between the residuals of the regression equations (26) . . . (49).
Taulukko 8. Regressioyhtälöiden (26) . . . (49) residuaalien väliset korrelaatiokertoimet.

	(27)	(28)	(29)	(30)	(31)	(32)	(33)	(34)	(35)	(36)	(37)	(38)	(39)	(40)	(41)	(42)	(43)	(44)	(45)	(46)	(47)	(48)	(49)
(26)	-.650	.066	.003	.042	-.093	.079	-.026	-.088	.055	.071	.084	-.103	-.034	-.041	.076	-.010	-.014	-.070	-.005	.059	.011	-.013	-.022
(27)		-.564	-.093	-.012	.088	.029	.049	.033	-.065	-.062	-.063	.041	.044	.063	-.000	-.025	-.039	.051	-.032	-.012	-.043	.062	-.009
(28)			-.344	-.152	-.056	-.093	.024	.044	.137	-.015	.006	-.003	-.024	.023	-.082	.023	.060	-.009	-.047	-.010	.039	-.027	.043
(29)				-.389	-.183	-.053	-.026	.066	-.107	.007	.081	.061	.006	-.087	.025	-.064	.050	-.031	.120	-.088	-.004	-.017	-.017
(30)					-.245	.035	-.172	-.157	.045	.071	-.028	-.114	.030	.033	.016	.023	-.072	.037	-.003	.107	-.053	.049	-.041
(31)						-.362	-.081	.019	-.073	-.079	-.037	.055	.030	-.019	-.022	.100	-.012	.043	.005	-.089	.075	-.059	.062
(32)							-.343	-.317	.004	.078	-.064	.007	-.047	-.023	.031	.011	-.003	-.039	.019	.110	-.116	-.006	.051
(33)								-.211	-.206	-.130	.083	-.036	-.054	.100	.032	-.064	.053	-.087	-.023	.014	.026	.077	-.115
(34)									-.248	-.078	-.141	.058	.053	-.010	-.061	.056	-.032	.082	-.036	-.073	.045	-.021	.006
(35)										-.289	-.106	-.092	-.042	-.006	-.012	.013	.015	.042	-.002	-.074	.115	-.149	.092
(36)											-.270	-.200	-.034	-.051	-.011	.007	.006	.017	-.049	.065	-.059	.137	-.152
(37)												-.255	-.124	-.096	.070	-.141	-.016	-.020	.087	.027	-.044	.071	-.016
(38)													-.355	-.103	-.086	.038	.001	.011	-.055	.044	.003	-.050	.037
(39)														-.193	-.211	-.067	.030	-.074	.053	-.051	.051	-.022	.052
(40)															-.332	-.112	-.127	-.017	.009	.074	-.084	.012	.008
(41)																-.277	-.132	-.026	-.043	-.062	.034	-.030	.067
(42)																	-.315	-.077	-.036	.034	.002	-.036	.014
(43)																		-.353	-.157	-.032	.080	-.025	-.021
(44)																			-.231	-.217	-.047	.045	-.028
(45)																				-.422	-.209	-.001	.077
(46)																					-.316	-.013	-.069
(47)																						-.451	-.073
(48)																							-.577

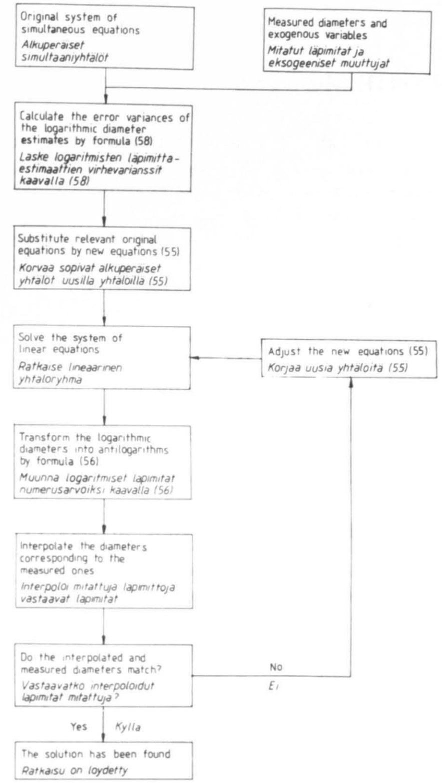

Fig. 2. Solution algorithm for a logarithmically linear taper curve model.
Kuva 2. Logaritmisesti lineaarisen runkokäyrämallin ratkaisualgoritmi.

The error variances of the logarithmic diameter estimates are derived from formula (58).

Secondly, each equation representing the diameter closest to the measured diameter is replaced by equation (55) where dl_{xh} equals the measured diameter.

The system of the logarithmically linear equations is solved and the corrections due to the error variance are made before the transformation of the endogenous variables into their antilogarithms.

New estimates for the measured diameters are calculated from the estimated relative-height diameters by the interpolation formulas. If the calculated diameters are close enough to the measured ones, the solution is accepted. Otherwise, the relative-height diameters closest to the measured diameters are corrected and the iteration is continued until a sufficient fit is attained.

The solution algorithm required less than one second of U1108 computer time per tree. The time could be further reduced by decreasing the number of the relative-height diameters as endogenous variables in the simultaneous equation model. On the other hand, an algorithm which directly solves the system of nonlinear equations (IMSL, Library 2, 1977) would require considerably more time.

4. APPLICATIONS OF THE MODELS

41. Taper curve

Relative-height diameter estimates derived by model I and their standard errors are presented in Appendix for various $d_{1.3}$–h combinations. Standard errors were derived from the following formula (see LAASASENAHO 1976):

$$s_{\hat{d}} = \sqrt{e^{v_i} - 1} \cdot \hat{d} \qquad (60)$$

where

$s_{\hat{d}}$ = standard error of the estimated value of the diameter
v_i = error variance of the logarithmic diameter estimate (formula 58)
$\hat{d}$ = estimated value of the diameter

Constant standard errors from Table 3 were applied in the estimation of v_i. Fig. 3 demonstrates some examples of these taper curves.

Slight irregularities in the taper curves are caused by the fact that the system of equations changes whenever the relative height of the measured diameter changes. Consequently, the taper curve does not change smoothly with regard to the height when the diameters are measured at absolute heights. This phenomenon may lead to erroneous diameter increment estimates if comparisons are made between the taper curves of the same tree in successive points of time.

Taper curves derived by model I with two measured diameters ($d_{1.3}$ and d_6) are demonstrated in Fig. 4. If the ratio between these diameters is exceptional, the taper curves are not quite smooth. This phenomenon is partly due to the fact that an exceptional $d_{1.3} - d_6$ combination often indicates true microlevel irregularities in the tree taper. Partly, this phenomenon is due to the fact that the simultaneous equation model does not take into account any logical considerations but shows only the statistical invariance between the diameters and other variables of the model. The relatively small number of sample trees (492), together with some exceptional trees (see Table 1), caused random variation in the parameters of the regression equations. This randomness shows up as uneven taper curves in Fig. 4.

Several ways to improve the taper curve models were considered. One possibility was to impose certain conditions to the differences between the successive diameters. Then, either one part of the data should have been rejected and the models would have been biased with respect to the whole data or nonlinear regression analysis should have been applied. The smoothness of the taper curves might be improved if the least significant independent variables were dropped from the regression equations. Since the awkwardnesses appeared only in very exceptional trees, the present models were accepted.

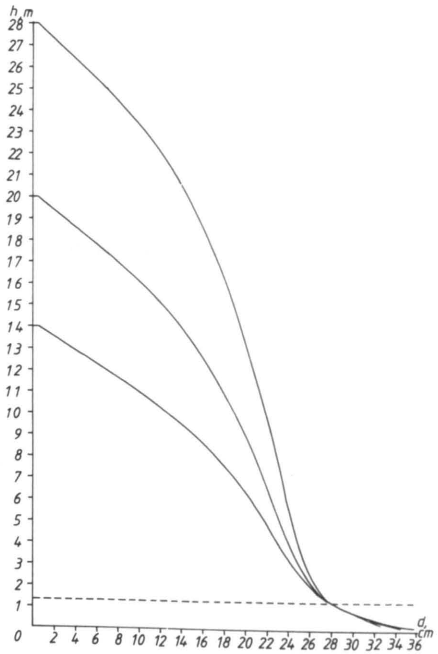

Fig. 3. Estimated taper curves when $d_{1.3}$ and height are assumed to be known.
Kuva 3. Estimoituja runkokäyriä, kun rinnankorkeusläpimitta ja pituus oletetaan tunnetuiksi.

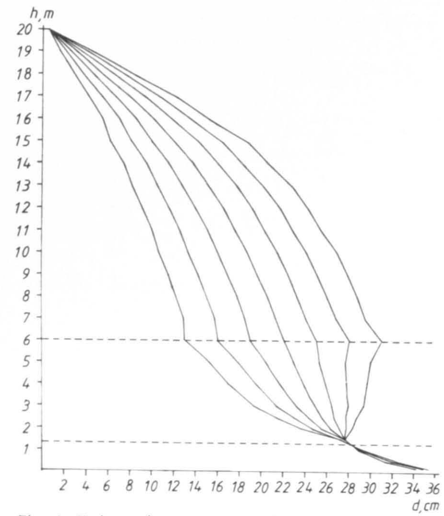

Fig. 4. Estimated taper curves when $d_{1.3}$, d_6, and h are assumed to be known.
Kuva 4. Estimoituja runkokäyriä, kun rinnankorkeusläpimitta, läpimitta 6 metrin korkeudelta ja pituus oletetaan tunnetuiksi.

The taper curve models may also be employed in the derivation of the upper diameter estimates at absolute heights. Fig. 5 shows the difference $d_{1.3}$–d_6 derived by model I as a function of $d_{1.3}$ and h. The results are similar to those presented by PÄIVINEN (1978). Only for large trees did the present model give somewhat smaller taperings.

Fig. 6 gives examples of the taper curves calculated by model II. It shows that the crown ratio has a marked influence upon the taper curve. The influence of the crown ratio is discernible only in the upper part of the stem. Thus, the measurement of the upper diameter (d_6) may not help the avoidance of biases due to exceptional crown ratios.

The use of variable error variances for the regression equations (formula 52) had little influence upon the taper curve. However, the estimated standard errors of the diameters (formula 58) were expectedly greater for short trees and smaller for tall trees than under the assumption of constant error variance.

42. Stem volume

Calculation of the stem volume or parts of it was accomplished by the numerical integration of the squared taper curve. If the taper curve is only an estimate of the true taper curve, the squared diameters have to be corrected before integration by the following formula:

$$E(d^2) = \bar{d}^2 + s_{\bar{d}}^2 \qquad (61)$$

where

$E(d^2)$ = expected value of the squared diameter
$\bar{d}$ = expected value of the diameter (formula 56)
$s_{\bar{d}}$ = standard error of the expected value of the diameter (formula 60)

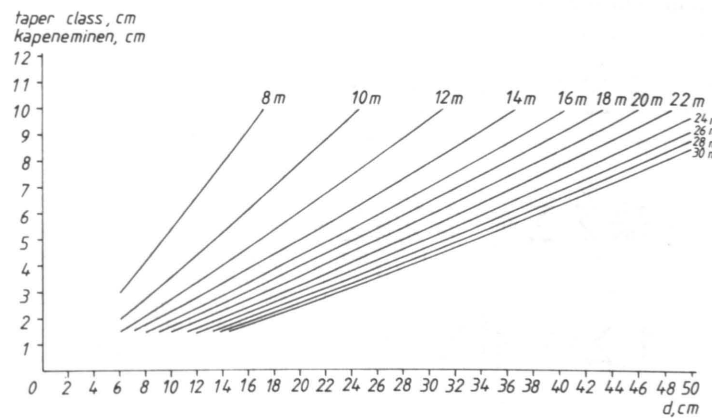

Fig. 5. Estimated taper class ($=d_{1.3}$–d_6) as a function of $d_{1.3}$ and h.
Kuva 5. Estimoitu kapeneminen rinnankorkeusläpimitan ja pituuden funktiona.

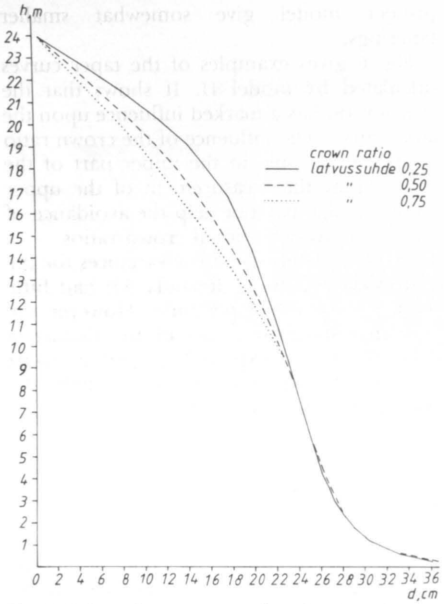

Fig. 6. Estimated taper curves when $d_{1.3}$, h, and crown ratio are assumed to be known.

Kuva 6. Estimoituja runkokäyriä, kun rinnankorkeusläpimitta, pituus ja latvussuhde oletetaan tunnetuiksi.

Formula (61) can be verified by the following proof.

In a given tree population the height of the trees is assumed to be constant (h). The taper curve of each tree i is referred as $f_i(x)$, where x equals the measurement height of each diameter. Then, the expected value of the mean stem volume is:

$$E(v) = \frac{\pi}{4m} \sum_{i=1}^{m} \int_0^h f_i^2(x) = \frac{\pi}{4m} \sum_{i=1}^{m} \frac{h}{n} \sum_{j=1}^{n} d_{ij}^2$$

$$= \frac{\pi h}{4mn} \sum_{i=1}^{m} \sum_{j=1}^{n} d_{ij}^2 = \frac{\pi h}{4n} \sum_{j=1}^{n} \frac{1}{m} \sum_{i=1}^{m} d_{ij}^2$$

$$= \frac{\pi h}{4n} \sum_{j=1}^{n} E(d_j^2) = \frac{\pi h}{4n} \sum_{j=1}^{n} (\hat{d}_j^2 + s_{\hat{d}_j}^2)$$

Here m indicates the number of trees and n the number of stem segments.

The standard errors of the intermediate diameters were interpolated linearly from the standard errors of the two closest relative-height diameters.

The step length in the numerical integration was 1/100 of the tree height. The volume of the stump was excluded from the volume of the stem. The stump height was assumed to be 1 per cent of the tree height.

Stem volumes based upon model I are given in Table 9 as a function of $d_{1.3}$ and h. The partial derivatives of the volume with respect to $d_{1.3}$ and h indicate that the taper curve model yields a satisfactorily continuous volume function with respect to $d_{1.3}$ and h.

Model I was also employed to derive stem volumes for trees with known $d_{1.3}$, d_6, and h. Examples of the estimated volumes are given in Table 10. The tapering $d_{1.3}$–d_6 is assumed to be 3 cm. Partial derivatives of the volume with respect to $d_{1.3}$ are quite satisfactory, but partial derivatives with respect to h reveal some undesirable features. A discontinuity of the volume function with respect to the height is discernible especially with large trees. The jumps in the derivatives are due to the changes in the system of equations with changing height of the tree (see p. 12).

Table 11 gives volume estimates derived by model II in which the crown ratio is an independent variable. There is a clear correlation between the crown ratio and the stem volume. With the exception of the shortest trees (outside Table 11) where the breast height is close to the top of the tree, the stem volume is greatest where the crown ratio is least.

For a tree having $d_{1.3}$ = 28 cm and h = 24 m, the volumes for the crown ratios .25, .50, and .75 are 731, 682, and 659 liters, respectively. The difference from the average volume 695 liters (Table 9) are +5.2, –2.0, and –5.2 per cent. Since the range of the crown ratio (from .032 to .906) in our sample trees (Table 2) exceeds that of the example, it is evident that the omission of the crown ratio may cause marked biases if only $d_{1.3}$ and h are measured.

The importance of the crown ratio was studied also in the case where both $d_{1.3}$ and d_6 were known. If $d_{1.3}$ = 28 cm, d_6 = 25 cm, and h = 24 m, and the crown ratios are .25, .50,

Table 9. Stem volume as a function of $d_{1.3}$ and h, litres.

Taulukko 9. Rungon tilavuus rinnankorkeusläpimitan ja pituuden funktiona, dm³.

$d_{1.3}$	h, m														
cm	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
2	1	1													
4	5	4	5	6											
6	12	8	10	13	16	19									
8	20	15	18	23	27	32	38	43							
10	31	23	29	36	42	50	57	66	74	83					
12		33	41	51	60	71	81	93	105	117	130	144			
14		44	55	69	81	95	109	124	140	156	174	192	210	229	
16		58	72	89	105	123	141	160	180	200	223	246	269	293	316
18			91	112	132	155	177	201	225	250	278	306	334	363	392
20			112	138	162	190	217	245	274	305	338	372	406	441	476
22			135	166	195	228	261	294	329	365	404	444	484	525	566
24				197	231	270	308	347	387	430	475	521	568	616	664
26				231	270	315	359	404	451	500	552	605	659	714	769
28				267	312	364	415	465	519	575	634	695	756	819	881
30					357	416	473	531	591	655	722	790	859	930	1000
32					405	471	536	600	668	740	814	891	969	1047	1125
34					456	530	603	674	750	830	912	998	1084	1171	1258
36						592	673	752	835	924	1016	1110	1205	1301	1397
38						658	747	834	926	1024	1124	1228	1332	1438	1543
40						727	825	919	1020	1128	1238	1351	1465	1581	1695
42							906	1009	1120	1237	1357	1480	1604	1730	1855
44							991	1103	1223	1351	1480	1615	1749	1885	2020
46							1080	1201	1331	1470	1609	1755	1900	2047	2193
48								1303	1443	1594	1744	1900	2057	2214	2371
50								1409	1559	1722	1883	2051	2219	2388	2556

Table 10. Stem volume as a function of $d_{1.3}$ and h with constant tapering ($d_{1.3}$ – d_6 = 3 cm), litres.

Taulukko 10. Rungon tilavuus rinnankorkeusläpimitan ja pituuden funktiona, kun kapeneminen ($d_{1.3}$–d_6) on 3 cm, dm³.

$d_{1.3}$	h, m					
cm	8	12	16	20	24	28
8	26	28	31			
12	63	74	85	99	114	
16	118	142	167	194	224	259
20	189	232	275	320	370	425
24	277	344	408	477	550	630
28	382	477	568	665	765	874
32		632	754	882	1014	1156
36		809	965	1130	1298	1475
40		1007	1202	1408	1615	1833

Table 11. Stem volume as a function of $d_{1.3}$, h, and crown ratio, litres.
a. Crown ratio .25
b. Crown ratio .50
c. Crown ratio .75
Taulukko 11. Rungon tilavuus rinnankorkeusläpimitan, pituuden ja latvussuhteen funktiona.
a. Latvussuhde 0.25
b. Latvussuhde 0.50
c. Latvussuhde 0.75

a.

$d_{1.3}$	h, m						
cm	4	8	12	16	20	24	28
4	4	6					
8	16	25	34	45			
12	35	55	75	97	122	150	
16	62	96	131	169	210	257	305
20		149	202	258	321	390	461
24		214	288	366	453	548	647
28		290	388	492	607	731	860
32			504	635	781	939	1102
36			634	796	977	1171	1371
40			778	974	1194	1426	1667

b.

$d_{1.3}$	h, m						
cm	4	8	12	16	20	24	28
4	4	6					
8	15	23	32	42			
12	33	51	70	91	113	139	
16	59	90	122	158	195	238	282
20		140	189	242	299	362	427
24		201	270	344	423	510	600
28		273	365	462	567	682	800
32			473	597	732	877	1026
36			596	749	916	1095	1277
40			733	918	1120	1335	1555

c.

$d_{1.3}$	h, m						
cm	4	8	12	16	20	24	28
4	4	6					
8	14	22	30	40			
12	32	49	67	88	109	134	
16	57	86	117	152	188	230	272
20		134	182	234	289	350	412
24		192	260	333	409	493	580
28		261	351	448	549	659	773
32			457	579	708	849	992
36			575	727	887	1060	1237
40			707	891	1085	1293	1507

and .75, as before, the stem volume estimates are 812, 752, and 715 liters, respectively. The differences from the average stem volume 765 liters (Table 10) are +6.1, −1.7, and −6.5 per cent. This result indicates that the measurement of the upper diameter cannot compensate for the missing crown ratio information (cf. Fig. 6), and the biases due to the crown ratio may even increase. However, if there is a correlation between d_6 and crown ratio, the measurement of d_6 may partially take care of the information the crown ratio yields. Therefore, it is improbable that the bias in the stem volume increases if d_6 is measured in addition to $d_{1.3}$.

When the constant standard errors of the original equations (1) . . . (24) and (26) . . . (49) were changed to standard errors which vary in accordance to the height of the tree (formula 52), the stem volumes changed only slightly. Table 12 gives new volume estimates which correspond to those in Table 9. The volumes are slightly higher for small trees and slightly lower for tall trees. This is mainly due to the change of the standard error in formula (61) and partly due to the change of the standard error in formula (56).

43. Timber assortments and stem value

The stems were scaled into different timber assortments in accordance with the following rules.

The minimum top diameter for saw logs was 160 mm and for pulpwood 70 mm over bark. The minimum length of the saw log part of the stem was 4.9 m. For non-saw timber trees the length of the pulpwood segment had to be at least 2 m. Otherwise the whole stem was counted as waste wood. For saw timber trees no constraint was imposed on the length of the pulpwood segment. The quality of the wood was not taken into account in the scaling of the stem into timber assortments. Neither was an optimal scaling procedure applied.

The probability that the stem belonged to the saw log category was assumed to be either 0 or 1. The outcome was determined by the dimensions of the expected taper curve.

Saw log and pulpwood percentages based upon model I and the constant error variances in Table 3 are presented in Table 13 as a function of $d_{1.3}$ and h.

To estimate the value of the stem it was assumed that the value of the saw log part of the stem depends upon the volume of the saw log part of the stem as follows (Metsätalouden . . . 1980):

Volume of the saw log part of the stem, m^3	Value of the saw log part of the stem, Fmk/m^3
0.2	88
0.3	98
0.4	104
0.5	110
0.6	114
0.7	116
0.8	117
0.9	118
1.0	119
1.1 +	120

The pulpwood was given the value of 50 Fmk/m^3. The stem values as a function of $d_{1.3}$ and h are given in Table 14 and the respective unit values of timber in Table 15.

Timber assortment percentages and stem values were also calculated by using the taper curve model II. Some examples of the results are given in Fig. 7 which demonstrates the dependence of the saw log percentage upon the crown ratio. If $d_{1.3} < 23$ cm the increase in crown ratio increases the saw log percentage, for larger trees the effect is opposite. The reason for this phenomenon can be seen in Fig. 6. The large crown ratio leads to small diameters in the upper part of the stem. In small trees this mainly decreases the pulpwood and waste wood part of the stem. In large trees the decrease occurs mainly in the saw log part of the stem.

For the sample tree with $d_{1.3} = 28$ cm and h = 24 m, the saw log percentages for crown ratios .25, .50, and .75 are 91.9, 89.7, and 88.5 and the stem values 80, 73, and 70 Fmk, respectively. The differences from the average value, 75 Fmk (Table 14) are +6.7, −2.7, and −6.7 per cent. The differences are slightly greater than those for the stem volumes due to the negative correlation between the saw

Table 12. Stem volume as a function of $d_{1.3}$ and h when the error variances of the regression equations (1) . . . (24) vary in accordance with h, litres.

Taulukko 12. Rungon tilavuus rinnankorkeusläpimitan ja pituuden funktiona, kun regressioyhtälöiden (1) . . . (24) jäännösvarianssit muuttuvat puun pituuden funktiona, dm³.

$d_{1.3}$ cm	h, m 4	8	12	16	20	24	28
4	4	6					
8	15	23	32	43			
12	33	51	71	93	116	144	
16	58	89	123	160	200	245	291
20		138	190	245	304	370	439
24		198	270	347	429	520	614
28		268	364	465	574	693	815
32			472	600	738	888	1043
36			593	751	923	1107	1296
40			727	919	1126	1347	1575

log percentage and the crown ratio. In smaller trees ($d_{1.3}$ < 23 cm) the influence of the crown ratio upon the stem value is not as marked because of the positive correlation between the saw log percentage and crown ratio (see Fig. 7).

The real influence of the crown ratio upon the stem value may be even greater than the figures above indicate since there is a negative correlation between the branchiness and quality of the wood. On the other hand, the live crown may not be a very good indicator of the dry branches which are of even greater importance to the wood quality than the living branches.

The employment of the varying error variance (formula 52) instead of the constant error variance in the taper curve estimation had little influence upon the timber assortment percentages and stem values, as could be expected from the results of the stem volumes (cf. Tables 9 and 12).

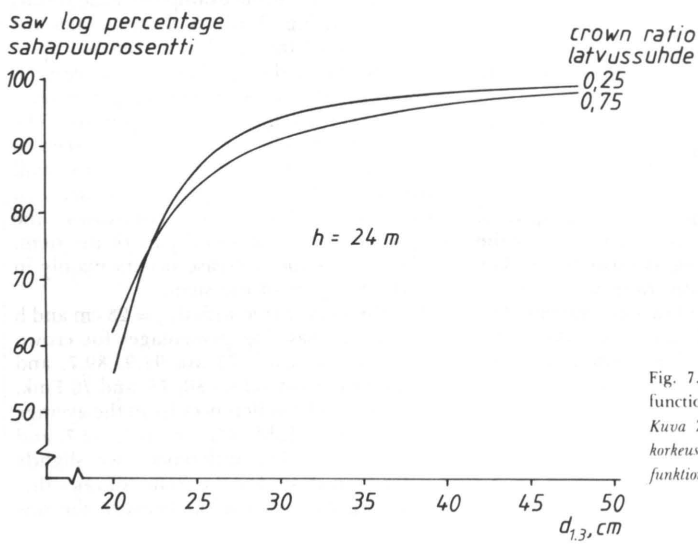

Fig. 7. Saw log percentage as a function of $d_{1.3}$ and crown ratio.
Kuva 7. Sahapuuprosentti rinnankorkeusläpimitan ja latvussuhteen funktiona.

Table 13. Timber assortment percentages of the stem as a function of $d_{1.3}$ and h.
a. Saw logs
b. Pulpwood

Taulukko 13. Rungon puutavaralajiprosentit rinnankorkeusläpimitan ja pituuden funktiona.
a. Tukkipuu
b. Kuitupuu

a.

$d_{1.3}$ cm	h, m 8	10	12	14	16	18	20	22	24	26	28	30
18	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	.0	30.7
20	.0	.0	.0	.0	59.2	58.3	57.5	58.0	57.4	58.0	57.4	58.1
22	.0	.0	75.9	74.9	72.9	72.0	72.4	71.7	72.1	72.6	73.1	73.6
24	.0	83.9	83.0	82.1	81.3	80.6	80.8	81.2	81.5	81.0	81.4	81.8
26	.0	88.2	87.3	86.6	86.7	86.1	86.3	86.6	86.1	86.4	86.8	86.3
28	.0	91.1	90.4	90.5	89.9	90.0	89.6	89.8	90.1	89.6	89.9	90.3
30		93.0	92.4	92.5	92.0	92.1	91.7	91.9	92.2	92.4	92.7	92.4
32		94.2	93.7	93.7	93.3	93.4	93.6	93.8	94.0	93.7	94.0	94.3
34		95.2	94.8	94.8	94.9	94.6	94.8	95.0	95.2	94.9	95.1	95.4
36			95.7	95.8	95.5	95.6	95.8	95.9	95.7	95.9	96.1	96.4
38			96.6	96.3	96.4	96.5	96.3	96.4	96.6	96.8	96.6	96.8
40			97.0	97.0	96.8	96.9	97.1	96.9	97.1	97.2	97.1	97.3
42				97.4	97.2	97.3	97.5	97.3	97.5	97.6	97.5	97.7
44				97.8	97.6	97.7	97.8	97.7	97.8	98.0	97.9	98.0
46				98.1	97.9	98.0	98.1	98.0	98.1	98.3	98.2	98.3
48					98.2	98.3	98.2	98.3	98.4	98.3	98.5	98.4
50					98.4	98.3	98.4	98.6	98.5	98.6	98.5	98.7

b.

$d_{1.3}$ cm	h, m 4	6	8	10	12	14	16	18	20	22	24	26	28	30
8	.0	66.2	59.2	52.2	49.5	47.0	45.9							
10	88.4	85.4	83.2	81.0	80.8	79.9	80.0	80.1	80.3					
12	93.4	92.0	91.1	90.9	90.2	90.2	90.3	91.1	91.3	91.5	92.4			
14	95.7	94.8	94.6	94.6	94.6	94.6	94.8	94.9	95.1	95.3	95.5	95.8	96.0	
16	97.1	96.5	96.5	96.5	96.5	96.6	96.7	96.9	97.0	96.8	97.0	97.2	97.4	97.3
18		97.5	97.5	97.6	97.6	97.7	97.6	97.7	97.9	98.0	97.9	98.0	98.2	67.4
20		98.3	98.1	98.2	98.3	98.4	39.1	40.1	41.0	40.3	41.2	40.7	41.2	40.7
22		98.7	98.6	98.7	22.9	23.8	25.9	26.6	26.4	27.2	26.7	26.4	26.0	25.4
24			99.0	15.0	16.1	17.0	17.7	18.5	18.2	18.0	17.6	18.2	17.9	17.5
26			99.2	11.0	11.9	12.7	12.5	13.2	12.9	12.7	13.2	13.0	12.6	13.1
28			99.3	8.2	9.0	8.9	9.5	9.3	9.9	9.6	9.4	9.9	9.6	9.4
30				6.4	7.1	7.0	7.4	7.3	7.7	7.6	7.4	7.1	6.9	7.2
32				5.4	5.9	5.9	6.3	6.2	6.0	5.8	5.6	6.0	5.7	5.4
34				4.4	4.9	4.8	4.7	5.0	4.9	4.7	4.5	4.8	4.5	4.4
36					3.9	3.9	4.2	4.0	3.9	3.7	4.0	3.8	3.6	3.4
38					3.2	3.4	3.3	3.2	3.4	3.3	3.1	3.0	3.2	2.9
40					2.8	2.7	3.0	2.8	2.7	2.9	2.7	2.6	2.7	2.6
42						2.4	2.6	2.5	2.3	2.5	2.4	2.2	2.3	2.2
44						2.0	2.2	2.1	2.0	2.2	2.0	1.8	2.0	1.8
46						1.7	2.0	1.8	1.7	1.9	1.7	1.6	1.7	1.5
48							1.7	1.5	1.7	1.6	1.4	1.6	1.4	1.5
50							1.4	1.6	1.4	1.3	1.4	1.3	1.4	1.3

Table 14. Stem value as a function of $d_{1.3}$ and h, Fmk.
Taulukko 14. Rungon arvo rinnankorkeusläpimitan ja pituuden funktiona, mk.

$d_{1.3}$	h, m													
cm	4	6	8	10	12	14	16	18	20	22	24	26	28	30
8	0	1	1	1	1	1	1							
10	1	1	1	2	2	2	3	3	3					
12	2	2	2	3	3	4	4	5	5	6	7			
14	2	3	3	4	4	5	6	7	7	8	9	10	11	
16	3	3	4	5	6	7	8	9	10	11	12	13	14	15
18		4	5	6	8	9	10	11	12	14	15	16	18	24
20		5	7	8	9	11	18	20	22	24	27	30	33	36
22		7	8	10	18	20	23	26	30	34	38	42	46	51
24			10	19	22	26	30	34	39	44	49	55	61	66
26			11	23	28	33	38	43	49	55	62	69	75	82
28			13	29	34	40	46	53	60	67	75	82	90	97
30				34	41	48	55	63	71	79	88	96	105	113
32				40	48	57	65	74	82	91	101	110	120	130
34				47	57	66	75	84	94	104	115	125	136	147
36					65	75	85	95	106	118	129	141	152	164
38					74	85	95	107	119	132	144	157	169	182
40					83	95	106	119	133	146	159	173	186	200
42						105	118	132	146	160	175	190	204	219
44						116	130	145	160	175	191	207	223	240
46						128	142	158	174	191	208	226	243	260
48							155	171	189	207	226	244	263	282
50							167	185	205	224	244	264	284	304

44. Increment

The stem is an outcome of the annual diameter and height increments. Therefore, a reliable prediction of the future taper curve would require increment models both for the diameters at various heights of the stem and for the height of the tree. Consequently, the derivation of the taper curve could be accomplished by the integration of the increment models over time. On the other hand, it is possible to find out increment estimates from the derivatives of the taper curve model. Even though this procedure is mathematically correct, the results are only rough estimates of the true increments.

The derivatives drawn from the taper curve model represent average increments and do not take into account all environmental factors. In the current taper curve model the whole history of the tree is integrated into the model. Thus, the separation of the influence of any changing environmental factors is difficult, if not impossible.

In the following discussion, two methods for deriving dimensional increments from the taper curve model are examined. The first method is based upon the employment of the Jacobian matrix. The Jacobian matrix contains the first derivatives of the equation with respect to all endogenous variables. The Jacobian matrix of the taper curve model describes the relations between the increments of the diameters and the height. The elements of the Jacobian matrix are derived from the following formulas:

$$\left.\begin{aligned} \frac{\partial d_i}{\partial d_j} &= \frac{a_{ij} d_i}{d_j} \qquad i = 1, \ldots, 24;\ j = 1, \ldots, 24 \\ \frac{\partial h}{\partial d_j} &= \frac{a_{25j} h}{d_j} \qquad j = 1, \ldots, 24 \\ \frac{\partial d_i}{\partial h} &= \frac{a_{i25} d_i}{h} + \frac{2 a_{i26} \ln(h) d_i}{h} \qquad i = 1, \ldots, 24 \end{aligned}\right\} \quad (62)$$

where a_{ij} = coefficient of the j^{th} variable in equation i (Tables 3 and 4)

The diagonal elements assume value −1.

Table 15. Timber value as a function of $d_{1.3}$ and h, Fmk/m³.
Taulukko 15. Runkopuun arvo rinnankorkeusläpimitan ja pituuden funktiona, mk/m³.

$d_{1.3}$	h, m													
cm	4	6	8	10	12	14	16	18	20	22	24	26	28	30
8	0	33	30	26	25	23	23							
10	44	43	42	41	40	40	40	40	40					
12	47	46	46	45	45	45	45	46	46	46	46			
14	48	47	47	47	47	47	47	47	48	48	48	48	48	
16	49	48	48	48	48	48	48	48	49	48	49	49	49	49
18		49	49	49	49	49	49	49	49	49	49	49	49	61
20		49	49	49	49	49	72	71	71	71	72	73	74	76
22		49	49	49	78	78	78	79	82	83	85	87	88	90
24			50	81	83	85	87	89	91	93	95	96	99	100
26			50	86	89	92	94	96	98	100	102	104	106	106
28			50	92	95	97	99	102	104	106	108	109	109	110
30				96	99	102	104	107	108	110	111	112	113	113
32				100	103	106	108	110	111	112	113	114	115	115
34				103	107	109	111	112	113	114	115	116	116	117
36					110	112	113	114	115	116	117	117	117	117
38					112	114	114	115	116	117	118	118	118	118
40					114	115	116	117	118	118	118	118	118	118
42						116	117	118	118	118	118	118	118	118
44						117	118	118	118	118	118	118	118	119
46						118	118	119	119	119	119	119	119	119
48							119	119	119	119	119	119	119	119
50							119	119	119	119	119	119	119	119

As with the original set of simultaneous equations, any row in the Jacobian matrix can be replaced by zeros and −1 on the diagonal if the increments of the respective diameter or height is known. For example, if the diameter increment $\Delta d1_i$ is known, the estimates of all the other diameter increments and the height increment can be obtained by solving the following matrix equation:

$$\begin{bmatrix} -1, \frac{\partial d_1}{\partial d_2}, \ldots & & \frac{\partial d_1}{\partial h} \\ \vdots & & \vdots \\ 0, \ldots, 0, -1, 0, \ldots, & & 0 \\ \vdots & & \vdots \\ \frac{\partial d_{24}}{\partial d_1}, \ldots & , \frac{\partial d_{24}}{\partial d_{23}}, -1, & \frac{\partial d_{24}}{\partial h} \\ \frac{\partial h}{\partial d_1}, \ldots & , \frac{\partial h}{\partial d_{24}}, & -1 \end{bmatrix} \begin{bmatrix} \Delta d_1 \\ \vdots \\ \Delta d_i \\ \vdots \\ \Delta d_{24} \\ \Delta h \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ -\Delta d1_i \\ \vdots \\ 0 \\ 0 \end{bmatrix} \quad (63)$$

When the Jacobian matrix method is applied, the solution may not give the increment of any relative-height diameter directly, because if the height of the tree increases, the measurement height changes. Consequently, a new taper curve has to be constructed after the solution to derive the real diameter increments at certain heights.

Fig. 8 gives some examples of the diameter increments with different $i_h/i_{d_{1.3}}$ ratios. The results are derived from model I. It is evident that the greater the height increment and the shorter the tree, the greater is the diameter increment in the upper part of the stem. In the lower part of the stem the differences are not so marked.

The Jacobian matrix approach was employed to derive average $i_h/i_{d_{1.3}}$ ratios (Table 16). These ratios may be used to estimate an unknown height increment. The ratios are evidently too low for small trees and too high for the tallest trees, since they reflect past rather than present growth.

The $i_h/i_{d_{1.3}}$ ratios in Table 16 were used to

Table 16. Average $i_h/i_{d_{1.3}}$ ratio as a function of $d_{1.3}$ and h.
Taulukko 16. Pituuskasvun ja rinnankorkeusläpimitan kasvun keskimääräinen suhde rinnankorkeusläpimitan ja pituuden funktiona.

$d_{1.3}$, cm	h, m: 2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
2	39	123													
4	20	61	96	127											
6	13	41	64	85	112	135									
8	10	30	47	64	83	101	119	138							
10	8	24	38	51	66	81	95	109	122	137					
12		20	31	43	55	67	79	90	101	114	128	141			
14		17	27	37	47	57	67	77	87	97	109	120	131	141	
16		15	23	32	41	50	58	67	75	85	95	105	114	122	130
18			21	29	36	44	52	59	67	75	84	93	101	108	115
20			18	26	32	40	46	53	60	68	76	83	90	97	103
22			17	23	29	36	42	48	55	61	69	75	82	88	93
24				21	27	33	38	44	50	56	63	69	75	80	85
26				20	25	30	35	40	46	52	58	63	69	74	79
28				18	23	28	33	37	43	48	53	59	64	68	73
30					21	26	31	35	40	45	50	55	59	64	68
32					20	25	29	33	37	42	47	51	55	60	63
34					19	23	27	31	35	39	44	48	52	56	60
36						22	25	29	33	37	41	45	49	53	56
38						21	24	27	31	35	39	43	46	50	53
40						20	23	26	30	33	37	41	44	47	50
42							22	25	28	32	35	39	42	45	48
44							21	23	27	30	34	37	40	43	46
46							20	22	26	29	32	35	38	41	44
48								21	25	28	31	34	36	39	42
50								21	24	27	29	32	35	37	40

Table 17. Average ratio (x100) between the basal area increment percentage and the volume increment percentage as a function of $d_{1.3}$ and h.
Taulukko 17. Pohjapinta-alan kasvuprosentin ja tilavuuskasvuprosentin keskimääräinen suhde (x100) rinnankorkeusläpimitan ja pituuden funktiona.

$d_{1.3}$, cm	h, m: 4	8	12	16	20	24	28
4	87	83					
8	87	83	79	77			
12		83	79	78	78	75	
16		82	79	79	78	76	76
20			80	79	78	76	76
24			80	80	78	77	77
28				80	78	77	77
32				81	79	77	78
36					79	78	78
40					79	78	78

Table 18. Ratio (x100) between the basal area increment percentage and the volume increment percentage as a function of $d_{1.3}$, h, and $i_h/i_{d_{1.3}}$.
Taulukko 18. Pohjapinta-alan kasvuprosentin ja tilavuuskasvuprosentin suhde (x100) rinnankorkeusläpimitan, pituuden sekä pituuskasvun ja läpimitan kasvun suhteen (= $i_h/i_{d_{1.3}}$) funktiona.

$d_{1.3}$, cm	h, m: 4	8	12	16	20	24	28
	$i_h/i_{d_{1.3}}$ = 100						
4	80	86					
8	67	75	78	82			
12		66	70	75	80	82	
16		59	64	70	75	77	79
20			59	65	70	72	75
24			55	61	65	68	71
28				58	61	64	68
32				55	58	61	65
36					55	58	62
40					53	56	60
	$i_h/i_{d_{1.3}}$ = 50						
4	89	94					
8	81	86	90	93			
12		80	84	88	92	94	
16		74	79	84	88	90	92
20			75	80	84	86	89
24			72	78	81	83	86
28				75	78	81	84
32				72	75	78	81
36					73	76	79
40					71	74	77
	$i_h/i_{d_{1.3}}$ = 0						
4	101	103					
8	101	102	104	106			
12		102	104	105	107	108	
16		102	103	105	106	108	109
20			103	105	106	108	109
24			103	105	106	107	109
28				105	106	107	109
32				105	106	107	108
36					106	107	108
40					106	107	108

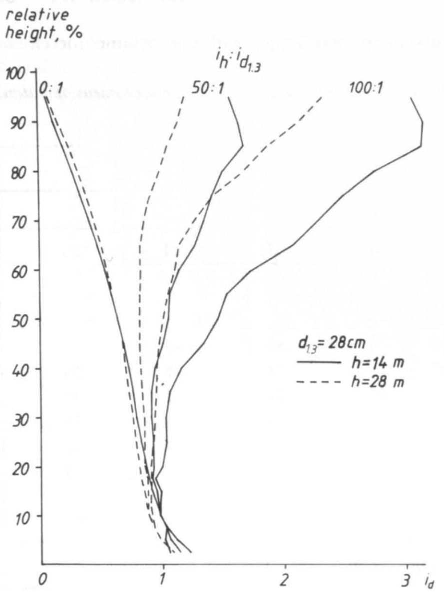

Fig. 8. Estimated relative diameter increments at different relative heights as a function of h and $i_h/i_{d_{1.3}}$.
Kuva 8. Suhteellisilta korkeuksilta estimoituja suhteellisia läpimitan kasvuja puun pituuden ja $i_h/i_{d_{1.3}}$ -suhteen funktiona.

derive ratios between the basal area increment percentage and the volume increment percentage (p_{i_g}/p_{i_v}). The results given in Table 17 indicate that the basal area increment percentage is usually 75 . . . 85 per cent of the volume increment percentage. ILVESSALO (1948, p. 17) suggests that the respective average figures vary from 65 to 75 per cent. Even though the procedure employed in this paper to derive the p_{i_g}/p_{i_v} ratios contains several weaknesses, the difference is so large that further studies with more appropriate data are necessary to explain the difference.

The Jacobian matrix method was also employed to derive p_{i_g}/p_{i_v} ratios for different i_h/i_d ratios. The results are given in Table 18. There is a clear negative correlation between the p_{i_g}/p_{i_v} and $i_h/i_{d_{1.3}}$ ratios. If the height increment equals zero, the p_{i_g}/p_{i_v} ratio invariably exceeds 1.

The Jacobian matrix approach is applicable only when very small increments are analyzed. Otherwise, the changes in the taper curve have to be derived by giving new increased stem dimensions to the model and by solving the system of equations with these new values. Then, however, it is better not to change the equation(s) to be substituted by the measured value(s) even though the relative height of the measured diameter may change due to the height increment (see p. 22).

45. ERROR OF THE VOLUME ESTIMATES

If the height of the tree is known the error variance formula of the stem volume estimate can be derived as follows:

$$s_{\hat{v}}^2 = E(v - \hat{v})^2 = E(v - E(v))^2$$

$$= E\left(\frac{\pi}{4}\sum_{j=1}^{n} p_j d_j^2 - \frac{\pi}{4} E\left(\sum_{j=1}^{n} p_j d_j^2\right)\right)^2$$

$$= \left(\frac{\pi}{4}\right)^2 \left(E\left(\sum_{j=1}^{n} p_j d_j^2\right)^2 - \left(E\left(\sum_{j=1}^{n} p_j d_j^2\right)\right)^2\right)$$

The first term of the expression in the brackets can be developed by the use of Taylor's expansion (see e.g. KILKKI 1979):

$$E\left(\sum_{j=1}^{n} p_j d_j^2\right)^2 = E\left(\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k d_j^2 d_k^2\right)$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j^2 \hat{d}_k^2 + \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j^2 s_{\hat{d}_k}^2$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_k^2 s_{\hat{d}_j}^2 + 4\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j \hat{d}_k \mathrm{cov}_{\hat{d}_j\hat{d}_k}^2$$

$$+ 2\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j M_{1\hat{d}_j^2\hat{d}_k} + 2\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_k M_{2\hat{d}_j^1\hat{d}_k}$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k M_{2\hat{d}_j^2\hat{d}_k}$$

where

$M_{1\hat{d}_j^2\hat{d}_k}$ = the central moment of the errors; of the first order with respect to d_j and of the second order with respect to d_k

$M_{2\hat{d}_j^2\hat{d}_k}$ = the central moment of the errors; of the second order with respect to d_j and d_k

The second term of the volume variance expression can be developed by employing formula (61):

$$\left(E\left(\sum_{j=1}^{n} p_j d_j^2\right)\right)^2 = \left(\sum_{j=1}^{n}\left(p_j\hat{d}_j^2 + p_j s_{\hat{d}_j}^2\right)\right)^2$$

$$= \left(\sum_{j=1}^{n} p_j\hat{d}_j^2 + \sum_{j=1}^{n} p_j s_{\hat{d}_j}^2\right)^2$$

$$= \left(\sum_{j=1}^{n} p_j\hat{d}_j^2\right)^2 + \left(2\sum_{j=1}^{n} p_j\hat{d}_j^2 \sum_{j=1}^{n} p_j s_{\hat{d}_j}^2\right)$$

$$+ \left(\sum_{j=1}^{n} p_j s_{\hat{d}_j}^2\right)^2$$

$$= \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j^2 \hat{d}_k^2 + 2\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j^2 s_{\hat{d}_k}^2$$

$$+ \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k s_{\hat{d}_j}^2 s_{\hat{d}_k}^2$$

When the second term is subtracted from the first term, some members of the terms cancel out and we get:

$$E\left(\sum_{j=1}^{n} p_j d_j^2\right)^2 - \left(E\left(\sum_{j=1}^{n} p_j d_j^2\right)\right)^2$$

$$= 4\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j \hat{d}_k \mathrm{cov}_{\hat{d}_j\hat{d}_k} + 2\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j M_{1\hat{d}_{j2}\hat{d}_k}$$

$$+ 2\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_k M_{2\hat{d}_j^1\hat{d}_k} + \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k M_{2\hat{d}_{j2}\hat{d}_k}$$

$$- \sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k s_{\hat{d}_j}^2 s_{\hat{d}_k}^2$$

If the diameter errors are small compared to the diameter estimates, the last four sums of the expression can be omitted and the error variance of the stem volume estimate can be written:

$$s_{\hat{v}}^2 = \frac{\pi^2}{4}\sum_{j=1}^{n}\sum_{k=1}^{n} p_j p_k \hat{d}_j \hat{d}_k \mathrm{cov}_{\hat{d}_j\hat{d}_k} \tag{64}$$

where

$s_{\hat{v}}^2$ = error variance of the stem volume estimate
p_j = length of the stem segment j
$\hat{d}_j$ = diameter estimate of the stem segment j
$\mathrm{cov}_{\hat{d}_j\hat{d}_k}$ = covariance of the errors of $\hat{d}_j$ and $\hat{d}_k$
n = number of stem segments

The stem was segmented in such a way that each relative-height diameter represented the middle diameter of the segment, except for the lowest segment that was represented by $d_{.01h}$. The covariance of the diameter errors was derived from the logarithmic covariance by the following formula which is similar to formula (60):

$$\mathrm{cov}_{\hat{d}_i\hat{d}_j} = (e^{\mathrm{cov}(\Delta_i\Delta_j)} - 1)\hat{d}_i\hat{d}_j \tag{65}$$

where $\mathrm{cov}(\Delta_i\Delta_j)$ comes from formula (57).

Formula (64) was employed to derive relative standard errors of the stem volume estimates. These figures corresponding to the volume estimates in Tables 9 and 12 are given in Tables 19 and 20, respectively. If it is assumed that the error variances of the logarithmic relative-height diameter equations are constant, the relative standard errors of the stem volume estimates generally increase with increasing height. This is due to the fact that the relative height of the breast height decreases with increasing height and the information value of the breast height diameter measurement with respect to the stem volume estimation also decreases. The only exceptions are short trees (h = 2 m) where the breast height diameter measurement occurs close to the top of the stem. Then, the information value of the breast height diameter decreases compared to taller trees.

If it is assumed that the error variances ot the original equations decrease with increasing height (formula 52), the relative standard error of the stem volume estimate does not increase as much with increasing height as in the previous case. For short trees

Table 19. Standard error of the stem volume estimated as a function of $d_{1.3}$ and h, per cent. Constant error variances are applied to regression equations (1) . . . (24).
Taulukko 19. Rinnankorkeusläpimitan ja pituuden funktiona estimoidun runkotilavuuden prosentuaalinen keskivirhe. Regressioyhtälöiden (1) . . . (24) virhevarianssit oletettu vakioiksi.

$d_{1.3}$ cm	h, m														
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
2	16.8	7.2													
4	16.9	6.9	7.8	8.0											
6	16.9	6.8	7.6	7.7	8.7	9.1									
8	17.0	6.8	7.4	7.5	8.4	8.9	9.2	9.5							
10	17.0	6.7	7.4	7.3	8.2	8.7	9.0	9.3	9.5	10.1					
12		6.7	7.3	7.2	8.1	8.5	8.8	9.1	9.4	10.0	10.6	10.8			
14		6.7	7.2	7.1	8.0	8.4	8.7	9.0	9.3	9.8	10.5	10.7	10.6	11.3	
16		6.7	7.2	7.0	7.9	8.3	8.6	8.9	9.1	9.7	10.4	10.6	10.5	11.2	11.6
18			7.1	7.0	7.8	8.2	8.5	8.8	9.0	9.6	10.3	10.5	10.4	11.1	11.5
20			7.1	6.9	7.8	8.2	8.5	8.7	9.0	9.5	10.2	10.4	10.3	11.0	11.4
22			7.1	6.9	7.7	8.1	8.4	8.7	8.9	9.5	10.1	10.3	10.2	10.9	11.3
24				6.9	7.7	8.0	8.3	8.6	8.8	9.4	10.0	10.2	10.1	10.8	11.2
26				6.8	7.6	8.0	8.3	8.6	8.8	9.3	10.0	10.2	10.0	10.8	11.1
28				6.8	7.6	7.9	8.2	8.5	8.7	9.3	9.9	10.1	10.0	10.7	11.1
30					7.5	7.9	8.2	8.5	8.7	9.2	9.9	10.1	9.9	10.7	11.0
32					7.5	7.9	8.1	8.4	8.6	9.2	9.8	10.0	9.9	10.6	11.0
34					7.5	7.8	8.1	8.4	8.6	9.2	9.8	10.0	9.8	10.6	10.9
36						7.8	8.1	8.3	8.5	9.1	9.7	9.9	9.8	10.5	10.9
38						7.8	8.0	8.3	8.5	9.1	9.7	9.9	9.8	10.5	10.9
40						7.7	8.0	8.3	8.5	9.0	9.7	9.9	9.7	10.5	10.8
42							8.0	8.2	8.4	9.0	9.6	9.8	9.7	10.4	10.8
44							8.0	8.2	8.4	9.0	9.6	9.8	9.6	10.4	10.8
46							7.9	8.2	8.4	9.0	9.6	9.8	9.6	10.4	10.7
48								8.2	8.4	8.9	9.5	9.7	9.6	10.3	10.7
50								8.1	8.3	8.9	9.5	9.7	9.5	10.3	10.7

(h = 2 m) the errors, quite expectedly, increase even more.

The relative standard error of the stem volume estimate decreases with increasing breast height diameter if the height remains constant. This is due to the fact that because the breast height diameter is assumed to be known exactly, the relative error of the stem volume decreases when the breast height diameter increases compared to the other diameters.

Next, the error variance of the saw log volume estimate is derived. Then it is necessary to take into account the variation of the length of the stem segment that belongs to this timber assortment. We have

$$s_{\hat{t}}^2 = \left(\frac{\pi}{4}\right)^2 \left(E\left(\sum_{j=1}^{m} p_j d_j^2\right)^2 - \left(E\left(\sum_{j=1}^{m} p_j d_j^2\right)\right)^2\right)$$

where

m = number of saw log segments

By omitting the central moments of the third and fourth order the first term in the brackets can be developed further:

$$E\left(\sum_{j=1}^{m} p_j d_j^2\right)^2 = E\left(\sum_{j=1}^{m}\sum_{k=1}^{m} p_j p_k d_j^2 d_k^2\right)$$

$$= \sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_j^2 \hat{d}_k^2 + \sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_k^2 s_{\hat{d}_j}^2$$

$$+ \sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_j^2 s_{\hat{d}_k}^2 + 4\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_j \hat{d}_k \mathrm{cov}_{\hat{d}_j \hat{d}_k}$$

Table 20. Standard error of the stem volume estimated as a function of $d_{1.3}$ and h, per cent. Error variances from formula (52) are applied to regression equations (1) . . . (24).
Taulukko 20. Rinnankorkeusläpimitan ja pituuden funktiona estimoidun runkotilavuuden prosentuaalinen keskivirhe. Regressioyhtälöiden (1) . . . (24) virhevarianssit laskettu kaavalla (52).

$d_{1.3}$ cm	h, m														
	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30
2	33.3	8.6													
4	33.6	8.3	9.1	9.0											
6	33.7	8.2	8.9	8.6	9.3	9.5									
8	33.8	8.1	8.7	8.4	9.1	9.2	9.1	9.1							
10	33.8	8.1	8.6	8.2	8.9	9.0	8.9	8.9	8.8	9.1					
12		8.1	8.5	8.1	8.7	8.8	8.8	8.7	8.7	8.9	9.3	9.4			
14		8.1	8.5	8.0	8.6	8.7	8.7	8.6	8.5	8.8	9.2	9.3	9.2	9.6	
16		8.1	8.4	7.9	8.5	8.6	8.6	8.5	8.4	8.7	9.1	9.1	9.1	9.5	9.7
18			8.4	7.9	8.4	8.5	8.5	8.4	8.4	8.6	9.0	9.1	9.0	9.4	9.6
20			8.3	7.8	8.4	8.5	8.4	8.4	8.3	8.5	8.9	9.0	8.9	9.3	9.6
22			8.3	7.8	8.3	8.4	8.3	8.3	8.2	8.5	8.8	8.9	8.8	9.3	9.5
24				7.7	8.3	8.3	8.3	8.2	8.1	8.4	8.8	8.8	8.8	9.2	9.4
26				7.7	8.2	8.3	8.2	8.2	8.1	8.3	8.7	8.8	8.7	9.1	9.4
28				7.7	8.2	8.2	8.2	8.1	8.0	8.3	8.7	8.7	8.6	9.1	9.3
30					8.1	8.2	8.1	8.1	8.0	8.2	8.6	8.7	8.6	9.0	9.3
32					8.1	8.2	8.1	8.0	8.0	8.2	8.6	8.6	8.6	9.0	9.2
34					8.1	8.1	8.1	8.0	7.9	8.2	8.5	8.6	8.5	8.9	9.2
36						8.1	8.0	8.0	7.9	8.1	8.5	8.6	8.5	8.9	9.1
38						8.1	8.0	7.9	7.8	8.1	8.5	8.5	8.4	8.9	9.1
40						8.0	8.0	7.9	7.8	8.1	8.4	8.5	8.4	8.8	9.1
42							7.9	7.9	7.8	8.0	8.4	8.5	8.4	8.8	9.0
44							7.9	7.8	7.8	8.0	8.4	8.4	8.3	8.8	9.0
46							7.9	7.8	7.7	8.0	8.3	8.4	8.3	8.8	9.0
48								7.8	7.7	7.9	8.3	8.4	8.3	8.7	8.9
50								7.8	7.7	7.9	8.3	8.4	8.3	8.7	8.9

$$+ \sum_{j=1}^{m}\sum_{k=1}^{m} \hat{d}_j^2 \hat{d}_k^2 \mathrm{cov}_{\hat{p}_j \hat{p}_k} + 2\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_k \hat{d}_j \hat{d}_k^2 \mathrm{cov}_{\hat{p}_j \hat{d}_j}$$

$$+ 2\sum_{j=1}^{m}\sum^{m} \hat{p}_k \hat{d}_j^2 \hat{d}_k \mathrm{cov}_{\hat{p}_j \hat{d}_k}$$

$$+ 2\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{d}_j \hat{d}_k^2 \mathrm{cov}_{\hat{p}_k \hat{d}_j}$$

$$+ 2\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{d}_j^2 \hat{d}_k \mathrm{cov}_{\hat{p}_k \hat{d}_k}$$

It can be assumed that the error in the length of the saw log part of the stem occurs only in the last segment. Then

$\mathrm{cov}_{\hat{p}_j \hat{p}_k} = 0$ for j=1,..., m-1 and k=1,..., m-1

$\mathrm{cov}_{\hat{p}_m \hat{p}_m} = s_{\hat{p}_m}^2$

$\mathrm{cov}_{\hat{p}_j \hat{d}_k} = 0$ for j=1,..., m-1

By combining some terms the previous expression can be rewritten:

$$\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_j^2 s_{\hat{d}_k}^2 + 2\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_k^2 s_{\hat{d}_j}^2$$

$$+ 4\sum_{j=1}^{m}\sum_{k=1}^{m} \hat{p}_j \hat{p}_k \hat{d}_j \hat{d}_k \mathrm{cov}_{\hat{d}_j \hat{d}_k} + \hat{d}_m^4 s_{\hat{p}_m}^2$$

$$+ 4\sum_{j=1}^{m} \hat{p}_j \hat{d}_j \hat{d}_m^2 \mathrm{cov}_{\hat{p}_m \hat{d}_j}$$

$$+\,4\sum_{j=1}^{m}\hat{p}_j\hat{d}_j^2\hat{d}_m\mathrm{cov}_{\hat{p}_m\hat{d}_m}$$

The second term of the error variance expression can be developed further:

$$\left(E\left(\sum_{j=1}^{m}p_jd_j^2\right)\right)^2=\left(\sum_{j=1}^{m}\hat{p}_j\hat{d}_j^2+2\sum_{j=1}^{m}\hat{d}_j\mathrm{cov}_{\hat{p}_j\hat{d}_j}\right.$$

$$\left.+\sum_{j=1}^{m}\hat{p}_js_{\hat{d}_j}^2\right)^2$$

$$=\left(\sum_{j=1}^{m}\hat{p}_j\hat{d}_j^2\right)^2+4\left(\sum_{j=1}^{m}\hat{p}_j\hat{d}_j^2\right)\left(\sum_{j=1}^{m}\hat{d}_j\mathrm{cov}_{\hat{p}_j\hat{d}_j}\right)$$

$$+\,2\left(\sum_{j=1}^{m}\hat{p}_j\hat{d}_j^2\right)\left(\sum_{j=1}^{m}\hat{p}_js_{\hat{d}_j}^2\right)+4\left(\sum_{j=1}^{m}\hat{d}_j\mathrm{cov}_{\hat{p}_j\hat{d}_j}\right)^2$$

$$+\,4\left(\sum_{j=1}^{m}\hat{d}_j\mathrm{cov}_{\hat{p}_j\hat{d}_j}\right)\left(\sum_{j=1}^{m}\hat{p}_js_{\hat{d}_j}^2\right)+\left(\sum_{j=1}^{m}\hat{p}_js_{\hat{d}_j}^2\right)^2$$

$$=\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_k\hat{d}_j^2\hat{d}_k^2+4\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{d}_j^2\hat{d}_k\mathrm{cov}_{\hat{p}_k\hat{d}_k}$$

$$+\,2\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_k\hat{d}_j^2s_{\hat{d}_k}^2+4\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{d}_j\hat{d}_k\mathrm{cov}_{\hat{p}_j\hat{d}_j}\mathrm{cov}_{\hat{p}_k\hat{d}_k}$$

$$+\,4\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_k\hat{d}_j\mathrm{cov}_{\hat{p}_j\hat{d}_j}s_{\hat{d}_k}^2+\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_ks_{\hat{d}_j}^2s_{\hat{d}_k}^2$$

By omitting the terms in which the errors are of the fourth order and by using the previous assumption of the covariances, the expression can be rewritten:

$$\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_k\hat{d}_j^2\hat{d}_k^2+4\sum_{j=1}^{m}\hat{p}_j\hat{d}_j^2\hat{d}_m\mathrm{cov}_{\hat{p}_m\hat{d}_m}$$

$$+\,2\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_k\hat{d}_j^2s_{\hat{d}_k}^2$$

The difference of the first and second terms of the error variance expression can now be written:

$$4\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_k\hat{d}_j\hat{d}_k\mathrm{cov}_{\hat{d}_j\hat{d}_k}+\hat{d}_m^4s_{\hat{p}_m}^2$$

$$+\,4\sum_{j=1}^{m}\hat{p}_j\hat{d}_j\hat{d}_m^2\mathrm{cov}_{\hat{p}_m\hat{d}_j}$$

Thus the error variance of the saw log volume estimate is:

$$s_{\hat{t}}^2=\left(\frac{\pi}{4}\right)^2\left(4\sum_{j=1}^{m}\sum_{k=1}^{m}\hat{p}_j\hat{p}_k\hat{d}_j\hat{d}_k\mathrm{cov}_{\hat{d}_j\hat{d}_k}+\hat{d}_m^4s_{\hat{p}_m}^2\right.$$

$$\left.+\,4\sum_{j=1}^{m}\hat{p}_j\hat{d}_j\hat{d}_m^2\mathrm{cov}_{\hat{p}_m\hat{d}_j}\right) \qquad (66)$$

where

$s_{\hat{t}}^2$ = error variance of the saw log volume estimate

$$s_{\hat{p}_m}=\frac{\hat{p}_m+\hat{p}_{m+1}}{\hat{d}_{m-1}-\hat{d}_{m+1}}s_{\hat{d}_m}$$

$$\mathrm{cov}_{\hat{p}_m\hat{d}_j}=\frac{\mathrm{cov}_{\hat{d}_m\hat{d}_j}}{s_{\hat{d}_m}}s_{\hat{p}_m}$$

It should be noticed that $\hat{p}_i$ assumes a constant value for all i=1, . . . , m.

The possibility that, due to the variation in the taper curve, the stem falls outside the saw timber category was not taken into account in the derivation of formula (66).

The relative standard errors of the saw log volume estimates derived by formula (66) are given in Tables 21 and 22. Table 21 shows the standard errors when the error variances of the logarithmic regression equations are assumed to be constant (Table 3) and Table 22 shows the standard errors when the error variances are in accordance with formula (52).

Table 21. Standard error of the saw log volume estimated as a function of $d_{1.3}$ and h, per cent. Constant error variances are applied to regression equations (1) . . . (24).

Taulukko 21. Rinnankorkeusläpimitan ja pituuden funktiona estimoidun tukkiosan tilavuuden prosentuaalinen keskivirhe. Regressioyhtälöiden (1) . . . (24) virhevarianssit oletettu vakioiksi.

$d_{1.3}$	h, m										
cm	10	12	14	16	18	20	22	24	26	28	30
18											42.2
20				21.9	23.2	25.6	28.2	29.4	26.1	28.4	29.7
22		13.1	14.3	15.6	16.6	18.2	20.0	20.9	21.2	23.2	24.5
24	11.0	12.2	13.2	14.1	15.0	16.3	17.9	16.3	16.3	17.7	18.5
26	10.3	11.2	12.0	12.7	13.3	14.4	14.4	14.9	14.9	16.1	16.8
28	9.7	10.4	11.0	11.7	12.2	13.1	14.2	13.6	13.7	14.9	15.7
30	9.6	10.3	10.0	10.6	11.1	11.9	12.9	13.4	13.5	13.5	14.2
32	8.8	9.4	9.9	10.5	10.9	11.8	11.9	12.3	12.4	13.4	14.0
34	8.7	9.3	9.8	9.8	10.1	10.9	11.8	12.2	12.2	13.2	12.7
36		8.8	9.2	9.7	10.1	10.8	11.7	12.1	11.2	12.1	12.6
38		8.7	9.2	9.6	10.0	10.7	10.9	11.2	11.2	12.0	12.5
40		8.7	9.1	9.5	9.4	10.1	10.8	11.1	11.1	12.0	12.5
42			8.7	9.1	9.4	10.0	10.8	11.1	11.0	11.3	11.7
44			8.7	9.0	9.3	10.0	10.7	11.0	10.4	11.2	11.7
46			8.6	9.0	9.3	9.9	10.6	10.4	10.3	11.2	11.6
48				8.9	9.2	9.5	10.1	10.4	10.3	11.1	11.5
50				8.9	8.8	9.4	10.1	10.4	10.2	11.1	11.5

Table 22. Standard error of the saw log volume estimated as a function of $d_{1.3}$ and h, per cent. Error variances from formula (52) are applied to regression equations (1) . . . (24).

Taulukko 22. Rinnankorkeusläpimitan ja pituuden funktiona estimoidun tukkiosan tilavuuden prosentuaalinen keskivirhe. Regressioyhtälöiden (1) . . . (24) virhevarianssit laskettu kaavalla (52).

$d_{1.3}$	h, m										
cm	10	12	14	16	18	20	22	24	26	28	30
18											34.3
20				20.8	21.3	22.7	24.5	25.2	21.2	22.4	23.0
22		13.6	14.2	14.9	15.3	16.3	17.6	18.2	18.6	19.9	20.7
24	11.9	12.6	13.0	13.4	13.6	14.3	15.2	13.6	13.6	14.4	14.8
26	11.2	11.7	11.9	12.1	12.1	12.6	12.3	12.5	12.5	13.2	13.7
28	10.4	10.7	10.9	11.0	11.1	11.5	12.1	11.8	11.9	12.7	13.3
30	10.3	10.6	9.9	10.1	10.2	10.7	11.3	11.6	11.8	11.4	11.8
32	9.4	9.7	9.8	10.0	10.1	10.5	10.4	10.6	10.6	11.2	11.6
34	9.4	9.6	9.7	9.3	9.4	9.7	10.3	10.5	10.5	11.1	10.6
36		9.1	9.2	9.3	9.3	9.6	10.2	10.4	9.7	10.2	10.5
38		9.1	9.1	9.2	9.2	9.6	9.5	9.6	9.6	10.1	10.4
40		9.0	9.1	9.1	8.7	9.0	9.4	9.5	9.5	10.0	10.3
42			8.7	8.7	8.6	8.9	9.4	9.5	9.5	10.0	9.8
44			8.6	8.6	8.6	8.9	9.3	9.4	9.0	9.5	9.8
46			8.6	8.6	8.5	8.8	9.3	9.0	9.0	9.5	9.7
48				8.5	8.5	8.8	8.9	9.0	8.9	9.4	9.7
50				8.5	8.1	8.4	8.8	8.9	8.9	9.4	9.6

5. RELIABILITY OF THE MODELS

The reliability of the simultaneous equation models was tested with the same data on which the models were based. An example of the measured and calculated taper curves is given in Fig. 9. The example shows the unevenness of the real taper curve and also the importance of the upper diameter (d_6) in the taper curve estimation.

The main results of the reliability tests are presented in Tables 23 and 24, which show both the absolute and relative standard errors and biases of the relative height diameters and stem volumes.

Table 23 gives the errors and biases for all trees when $d_{1.3}$ and h assume measured values. The results are calculated using both models I and II. Also a model with the stand numbers as dummy variables was employed (see p. 9).

The calculated absolute standard errors of the diameter estimates are close to the standard errors of the mean tree of the sample trees ($d_{1.3} \cong 24$ cm, $h \cong 18$ m) (see Appendix). The biases in the diameter estimates are insignificant. This is especially true with regard to the relative biases which are of greater importance since the models are logarithmic. The existing biases are due to the fact that the measured breast height diameters do not perfectly coincide with the values interpolated from the measured relative-height diameters, but there are small biases. If a relative-height diameter had been employed instead of the breast height diameter, no biases would have occurred.

The measurement of the crown ratio reduces the standard errors of the diameters which are above 50 per cent of the tree height. The inclusion of the dummy variables in addition to the crown ratio reduces the standard errors of all diameter estimates. Also tested was whether the crown ratio is needed at all if the stand numbers are taken as a dummy variables. The results showed that the importance of the crown ratio as a taper curve predictor decreases but does not disappear if the influence of the stand is eliminated.

The relative standard error of the stem volume estimate is reduced from 8.1 per cent to 7.4 per cent if the crown ratio is known in addition to $d_{1.3}$ and h. The standard error is further reduced to 6.9 per cent if the variation between the stands is eliminated by the use of the dummy variables. The employment of the crown ratio does not reduce the absolute standard error of the volume estimate. This phenomenon is somewhat surprising since the absolute standard errors of the diameters are markedly reduced in the upper part of the stem by the employment of the crown ratio as a measured value. No marked relative biases are discernible in the volumes. The slight absolute biases in the volumes are due to the fact that constant relative variances of the diameter estimates were applied to volume estimation. Variable variances (formula 52) would have been in better correspondence with the data.

Table 24 gives the respective figures for trees taller than 8.6 meters.The standard errors of the diameters and volumes do not deviate markedly from the results of the whole data. The biases in the relative values are due to the fact that only one part of the original data is employed in the examination.

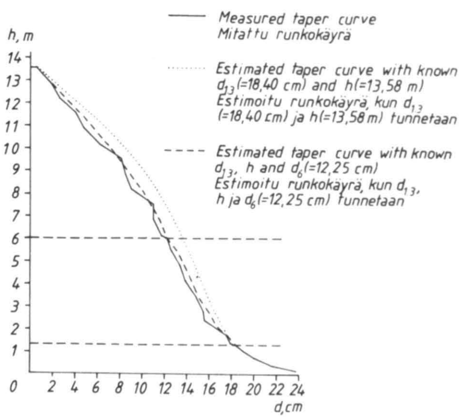

Fig. 9. Example of the differences between the measured taper curve and estimated taper curves.

Kuva 9. Esimerkki mitatun runkokäyrän ja estimoitujen runkokäyrien eroista.

Table 23. Standard errors (s) and biases (b) of the estimated relative-height diameters and volumes with various measurement combinations. All sample trees.

Taulukko 23. Kaikkien koepuiden suhteellisilta korkeuksilta estimoitujen läpimittojen ja tilavuuksien keskivirheet (s) ja harhat (b) eri mittaustietoja käytettäessä.

Diameter *Läpimitta*	Measured information – *Mittaustiedot*											
	$d_{1.3}$, h				$d_{1.3}$, h, crown ratio				$d_{1.3}$, h, crown ratio, stand number			
	mm (l)		per cent, %		mm (l)		per cent, %		mm (l)		per cent, %	
	s	b	s	b	s	b	s	b	s	b	s	b
$d_{.01h}$	17.0	–.3	5.9	–.1	17.0	–.3	5.9	–.1	15.7	–.2	5.3	–.1
$d_{.025h}$	11.7	–.4	4.7	–.1	11.6	–.4	4.7	–.1	10.6	–.3	4.2	–.1
$d_{.05h}$	7.3	–.3	3.6	–.1	7.3	–.4	3.5	–.1	7.0	–.3	3.3	–.1
$d_{.075h}$	5.6	–.2	2.9	–.1	5.6	–.2	2.9	–.1	5.4	–.2	2.8	–.1
$d_{.1h}$	6.5	–.1	2.9	–.1	6.4	–.1	2.8	–.1	6.1	–.1	2.7	–.1
$d_{.125h}$	7.6	+.0	3.2	–.1	7.6	+.0	3.2	–.1	7.1	–.0	3.0	–.1
$d_{.15h}$	8.0	+.0	3.3	–.1	8.0	+.0	3.3	–.1	7.4	–.0	3.1	–.1
$d_{.175h}$	8.9	+.1	3.7	–.1	8.9	+.1	3.6	–.1	8.1	+.0	3.4	–.1
$d_{.2h}$	9.2	+.1	3.8	–.1	9.2	+.1	3.8	–.1	8.4	+.0	3.5	–.1
$d_{.25h}$	9.7	+.2	4.2	–.1	9.6	+.1	4.2	–.1	8.9	+.1	3.9	–.1
$d_{.3h}$	9.9	+.2	4.6	–.1	9.8	+.2	4.6	–.1	9.0	+.1	4.3	–.1
$d_{.35h}$	10.1	+.2	4.9	–.1	10.1	+.2	4.9	–.1	9.1	+.1	4.5	–.1
$d_{.4h}$	10.6	+.2	5.4	–.1	10.5	+.1	5.3	–.1	9.5	+.0	4.9	–.1
$d_{.45h}$	11.4	+.2	6.1	–.1	11.3	+.1	6.0	–.1	10.2	+.0	5.5	–.1
$d_{.5h}$	12.1	+.2	7.0	–.0	11.9	+.1	6.7	–.1	10.9	+.0	6.3	–.1
$d_{.55h}$	12.7	+.2	7.7	–.0	11.9	+.1	7.0	–.1	10.8	+.0	6.5	–.1
$d_{.6h}$	14.1	+.3	9.4	–.0	12.7	+.0	8.1	–.1	11.4	+.0	7.4	–.0
$d_{.65h}$	15.5	+.4	11.2	–.0	13.4	+.1	9.2	–.0	12.4	+.0	8.6	–.0
$d_{.7h}$	16.6	+.5	13.7	–.0	14.0	+.1	10.6	–.0	12.4	+.1	9.8	–.0
$d_{.75h}$	16.9	+.5	16.2	+.0	14.3	+.2	12.5	–.0	13.1	+.2	11.7	+.0
$d_{.8h}$	17.2	+.5	19.6	+.0	13.8	+.1	14.1	+.0	12.5	+.1	13.0	+.1
$d_{.85h}$	16.7	+.5	24.7	–.0	13.5	+.2	17.3	+.1	12.6	+.2	16.4	+.2
$d_{.9h}$	14.5	+.3	31.6	–.2	12.0	+.1	22.0	+.1	11.2	+.2	20.7	+.2
$d_{.95h}$	11.2	–.0	42.1	–.6	9.8	–.1	31.3	–.1	9.3	+.0	28.7	+.1
Volume *Tilavuus*	63.3	+7.4	8.1	–.0	63.5	+5.5	7.4	–.1	55.3	+3.6	6.9	–.0

Table 24 gives also standard errors and biases for trees taller than 8.6 meters when the diameter at 6 meters' height assumes the measured value. Above the 10 per cent mark the standard errors of the diameter estimates are clearly smaller than in the case where only $d_{1.3}$ was assumed to be known. The importance of the crown ratio becomes evident above the 55 per cent mark.

Both the crown ratio and the variation between the stands bear a marked influence upon the relative standard error of the stem volume estimate even though the upper diameter (d_6) is fixed. This result indicates that biased results may be due to the common measurement practice in Finland where two diameters ($d_{1.3}$ and d_6) are measured from the sample trees.

Small biases in the diameter and volume estimates due to the slightly biased regression equations (see p. 10) were detected with respect to the size of the tree.

Table 24. Standard errors (s) and biases (b) of the estimated relative-height diameters and volumes with various measurement combinations. Trees taller than 8.6 meters.
Taulukko 24. Yli 8.6 metriä pitkien puiden suhteellisilta korkeuksilta estimoitujen läpimittojen ja tilavuuksien keskivirheet (s) ja harhat (b) eri mittaustietoja käytettäessä.

Diameter *Läpimitta*	Measured information – *Mitatut tiedot*											
	$d_{1.3}$, h				$d_{1.3}$, h, crown ratio				$d_{1.3}$, h, crown ratio, stand number			
	mm (l)		per cent, %		mm (l)		per cent, %		mm (l)		per cent, %	
	s	b	s	b	s	b	s	b	s	b	s	b
$d_{.01}h$	17.4	–.7	5.6	–.3	17.4	–.6	5.6	–.3	16.1	–.4	5.1	–.3
$d_{.025}h$	11.5	–.9	4.2	–.3	11.5	–.9	4.2	–.4	10.6	–.6	3.8	–.3
$d_{.05}h$	6.9	–.7	3.0	–.3	7.0	–.7	3.0	–.3	6.7	–.6	2.8	–.3
$d_{.075}h$	5.3	–.5	2.3	–.3	5.3	–.5	2.3	–.3	5.1	–.4	2.2	–.3
$d_{.1}h$	6.4	–.2	2.3	–.2	6.4	–.2	2.3	–.3	6.1	–.2	2.2	–.2
$d_{.125}h$	7.8	–.2	2.7	–.3	7.8	–.2	2.8	–.3	7.3	–.2	2.6	–.3
$d_{.15}h$	8.4	–.1	3.1	–.2	8.3	–.1	3.1	–.2	7.7	–.1	2.9	–.2
$d_{.175}h$	9.3	+.0	3.4	–.2	9.3	+.0	3.4	–.2	8.5	–.1	3.2	–.2
$d_{.2}h$	9.7	–.0	3.7	–.2	9.6	–.0	3.7	–.2	8.9	–.1	3.4	–.2
$d_{.25}h$	10.2	+.0	4.1	–.3	10.1	+.0	4.1	–.3	9.3	–.1	3.8	–.2
$d_{.3}h$	10.3	+.0	4.5	–.3	10.3	+.0	4.5	–.3	9.4	–.1	4.2	–.3
$d_{.35}h$	10.6	+.1	4.8	–.3	10.6	+.1	4.8	–.3	9.6	–.1	4.4	–.3
$d_{.4}h$	10.9	+.1	5.1	–.3	10.9	+.0	5.1	–.3	9.9	–.1	4.7	–.3
$d_{.45}h$	11.7	+.2	5.8	–.3	11.6	+.1	5.7	–.3	10.5	–.1	5.3	–.3
$d_{.5}h$	12.5	+.0	6.6	–.3	12.3	–.1	6.4	–.3	11.2	–.1	5.9	–.3
$d_{.55}h$	12.9	+.1	7.2	–.3	12.3	–.1	6.7	–.3	11.1	–.1	6.1	–.3
$d_{.6}h$	14.4	+.2	8.7	–.3	13.1	–.1	7.7	–.3	11.8	–.1	7.0	–.3
$d_{.65}h$	15.8	+.3	10.2	–.4	13.8	–.1	8.6	–.4	12.7	–.1	8.0	–.3
$d_{.7}h$	16.9	+.5	12.5	–.3	14.4	+.1	10.0	–.3	12.7	–.0	8.9	–.3
$d_{.75}h$	17.3	+.6	15.0	–.3	14.8	+.1	11.9	–.3	13.5	+.1	11.0	–.2
$d_{.8}h$	17.5	+.6	18.1	–.3	14.3	+.1	13.4	–.3	12.9	+.1	12.0	–.2
$d_{.85}h$	16.7	+.7	22.2	–.3	13.8	+.2	16.2	–.1	12.8	+.2	15.0	–.0
$d_{.9}h$	14.4	+.5	28.4	–.3	12.2	+.2	20.8	–.1	11.3	+.3	19.0	+.1
$d_{.95}h$	11.1	+.1	40.4	–.7	9.9	+.0	30.6	–.1	9.4	+.1	27.4	+.2
Volume *Tilavuus*	67.1	+8.3	7.8	–.4	67.4	+6.1	7.2	–.4	58.6	+4.0	6.6	–.4

Table 24. Continued.
Taulukko 24. Jatkoa.

Measured information – *Mitatut tiedot*											
$d_{1.3}$, d_6, h				$d_{1.3}$, d_6, h crown ratio				$d_{1.3}$, d_6, h, crown ratio, stand number			
mm (l)		per cent, %		mm (l)		per cent, %		mm (l)		per cent, %	
s	b	s	b	s	b	s	b	s	b	s	b
17.4	–.8	5.6	–.4	17.5	–.8	5.6	–.4	16.1	–.5	5.1	–.3
11.5	–1.1	4.2	–.4	11.5	–1.0	4.2	–.4	10.6	–.7	3.8	–.3
6.9	–.9	3.0	–.4	6.9	–.9	3.0	–.4	6.7	–.7	2.8	–.3
5.0	–.5	2.2	–.3	5.0	–.6	2.3	–.3	4.9	–.5	2.1	–.3
4.8	–.3	2.0	–.2	4.8	–.3	2.0	–.2	4.8	–.2	1.9	–.2
5.6	–.1	2.2	–.1	5.6	–.1	2.2	–.1	5.4	–.1	2.1	–.2
5.6	+.0	2.3	–.0	5.6	+.0	2.3	–.0	5.4	–.0	2.2	–.1
5.6	+.1	2.3	–.0	5.6	+.1	2.3	–.0	5.4	+.1	2.3	–.1
5.8	+.1	2.5	+.0	5.7	+.1	2.4	–.0	5.4	+.1	2.3	–.0
5.1	+.2	2.5	+.1	5.0	+.2	2.4	+.0	4.9	+.1	2.3	–.0
5.6	+.3	2.7	+.1	5.5	+.3	2.6	+.1	5.2	+.2	2.5	+.0
6.3	+.4	2.9	+.1	6.2	+.3	2.8	+.1	5.9	+.2	2.7	+.0
6.5	+.4	3.1	+.1	6.5	+.3	3.0	+.1	6.2	+.2	2.9	+.0
7.2	+.5	3.4	+.2	7.2	+.3	3.4	+.1	6.6	+.2	3.2	+.1
8.2	+.4	4.0	+.2	8.2	+.2	4.0	+.1	7.5	+.2	3.6	+.1
9.2	+.5	4.6	+.1	8.7	+.2	4.4	+.1	7.9	+.2	3.9	+.1
11.2	+.6	6.0	+.2	10.0	+.2	5.4	+.1	8.9	+.2	4.8	+.1
12.6	+.7	7.5	+.1	10.7	+.2	6.2	+.0	9.9	+.2	5.8	+.1
14.2	+.9	9.6	+.2	11.9	+.3	7.8	+.1	10.4	+.3	6.9	+.1
15.3	+.9	12.3	+.2	12.9	+.4	10.0	+.1	11.8	+.4	9.2	+.2
16.0	+.9	15.5	+.2	13.0	+.3	12.0	+.1	11.7	+.4	10.6	+.2
15.6	+1.0	19.4	+.3	13.0	+.4	15.0	+.2	12.1	+.5	13.8	+.4
13.7	+.7	25.4	+.3	12.0	+.3	20.0	+.3	11.1	+.5	18.2	+.5
10.7	+.3	36.6	+.1	9.9	+.1	30.0	+.2	9.4	+.3	26.5	+.7
38.7	+5.7	4.4	+.1	37.2	+3.2	4.0	+.1	33.0	+2.6	3.7	+.1

6. DISCUSSION

The results of this paper verify the initial assumption that the simultaneous equation approach is applicable to the derivation of taper curve models. Any predicting variable can be easily included in the simultaneous models. The model for Scots pine also proved to be quite simple. This is an important factor with regard to the costs of computation.

With more comprehensive data and more efficient computers it might be possible to employ even larger and more complex models than those presented in this paper. In the new models, any variable that can be measured from the tree would be considered as an endogenous variable. Only the environmental factors would remain exogenous variables. Then, variables describing the crown and the bark, for example, might be new endogenous variables. It is possible that the bark thickness at breast height, for instance, might be a useful endogenous variable (cf. NÄSLUND 1947). The employment of the weighted regression analysis should also be studied in the estimation of the model parameters (see p. 10).

The implementation of new endogenous variables might lead to models in which the solution of the system of simultaneous equations would require the employment of nonlinear solution algorithms (see e.g. IMSL, Library 2, 1977). It is also possible that the use of the present method for other tree species than Scots pine would call for nonlinear solution algorithms.

Major improvements in the present static taper curve models look improbable. The stem form is a result from the lifelong influence of the environment. Consequently, the influence of the stand density, for example, cannot be easily taken into account in static models where only the stand density of the few previous years is known. Dynamic models are therefore required.

The endogenous variables in the dynamic models should comprise the incremental changes of the tree characteristics in addition to the characteristics themselves. It would then be necessary to include all measurable parts of the tree and their increments during a certain period of time in the set of the endogenous variables. Then, the measured environment would have a meaningful linkage to the increment of the tree. This new approach would then permit testing of the validity of the traditional stem form theories.

A Fortran program based upon models I and II was written for VAX 11/780 computer to calculate taper curves, stem volumes, timber assortment percentages, and stem values. The program gives also standard errors of the diameter, stem volume, and saw log volume estimates. Up to 20 diameters measured at arbitrary heights of the stem, the height of the tree, and the crown ratio may be utilised in the determination of the taper curve. The program is available from the authors.

7. SUMMARY

Taper curve models based upon simultaneous equations were derived. The data consisted of 492 Scots pines from Southern Finland. Two systems of simultaneous equations were constructed, one without the crown ratio and the other with the crown ratio as an exogenous variable. The endogenous variables consisted of 24 relative-height diameters and the height of the tree. The parameters of the model were derived by the ordinary least squares method.

In most applications, the height of the tree was exogenised. The logarithmically linear relationships between the relative-height diameters were utilised in the solution algorithm. The algorithm included both standard matrix operations and an iterative part in which the taper curve was fitted to any measured diameters by the natural cubic spline interpolation formula.

The models were applied to the derivation of taper curves, stem volumes, timber assortment percentages, and stem values. An experiment was also made to derive diameter and height increments from the taper curve model. Formulas for the standard errors of the stem volume and saw log volume estimates were also derived.

The reliability of the models was tested on the original data. Slight biases in the models were detected with respect to the size of the tree.

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SELOSTE

MÄNNYN RUNKOKÄYRÄMALLEJA JA NIIDEN SOVELLUTUKSIA

Simultaanisen moniyhtälömallin käyttö runkokäyrän estimoinnissa on esitetty aiemmissa tutkimuksissa (KILKKI ym. 1978, KILKKI ja VARMOLA 1979). Näissä tutkimuksissa aineistot ovat kuitenkin olleet joko pieniä tai epäedustavia. Tämän tutkimuksen tarkoituksena on kehittää riittävän suuren ja edustavan aineiston avulla simultaaninen moniyhtälömalli männyn runkokäyrälle ja esitellä mallin käyttö runkotilavuuden ja puutavaralajien osuuksien määrittämiseen sekä tarkastella mallin soveltuvuutta kasvun estimointiin.

Simultaanisella moniyhtälömallilla puun eri tunnusten (endogeenisten muuttujien) ja puun ympäristöä kuvaavien tunnusten (eksogeenisten muuttujien) väliset relaatiot kuvataan niin monella yhtälöllä kuin mallissa on endogeenisia muuttujia. Yhtälöiden parametrit estimoidaan pienimmän neliösumman menetelmällä siten, että kukin endogeeninen muuttuja on vuorollaan selitettävänä muuttujana.

Mikä tahansa puun tunnus voidaan ottaa eksogeeniseksi muuttujaksi, jos se tunnetaan aina mallin sovellutustilanteessa. Samoin voidaan mikä tahansa mallin endogeeninen muuttuja sovellutustilanteessa eksogenisoida sijoittamalla endogeenisen muuttujan mitattu arvo kaikkiin mallin yhtälöihin. Tällöin se yhtälö, jossa endogeeninen muuttuja on selitettävänä muuttujana, korvataan yhtälöllä, joka ilmaisee endogeenisen muuttujan ja sen mitatun arvon yhtäsuuruuden.

Perusjoukkona ovat Etelä- ja Keski-Suomen männyt. Aineisto kerättiin enimmäkseen päätehakkuualoilta. Harvennusmetsiköissä puut valittiin ajourilta. Yhdestä metsiköstä kelpuutettiin enintään 20 puuta, joiden tuli olla eri läpimitta–pituusluokissa (luokkaväli 2 cm – 2 m). Kuhunkin luokkaan hyväksyttiin kaikkiaan vain viisi puuta. Näin vältettiin aineiston kasautuminen tavanomaisimpiin luokkiin. Yhteensä 495 puuta mitattiin (taulukko 1 ja kuva 1). Puista mitattiin ristikkäin millimetrin tarkkuudella läpimitat seuraavilta prosenttikorkeuksilta: 1, 2.5, 5, 7.5, 10, 12.5, 15, 17.5, 20, 25, 30, 35, 40, 45, 50, 55, 60, 65, 70, 75, 80, 85, 90 ja 95. Lisäksi mitattiin $d_{1.3}$, d_6, h, latvusraja, kantoikä ja normaalit metsikkötiedot.

Aiemmissa tutkimuksissa endogeenisina eli simultaanisessa moniyhtälömallissa ratkaistavina muuttujina olivat osakorkeusläpimitat; pituus oli eksogeeninen eli mitattu muuttuja. Tässä tutkimuksessa myös pituus on periaatteessa endogeeninen muuttuja. Erilaisia puuta ja metsikköä kuvaavia tunnuksia testattiin eksogeenisina muuttujina. Näistä latvussuhde hyväksyttiin eksogeeniseksi muuttujaksi.

Taulukoissa 3 ja 4 on esitetty kaksi simultaanista moniyhtälömallia I ja II (yhtälöt 1 . . . 25 ja 26 . . . 50). Selitettävinä muuttujina ovat osakorkeusläpimittojen logaritmit ja pituuden logaritmi. Selittävinä muuttujina ovat mallissa I muiden osakorkeusläpimittojen logaritmit, pituuden logaritmi ja pituuden logaritmin neliö sekä mallissa II myös latvussuhteen logaritmi ja latvussuhteen logaritmin neliö.

Regressioyhtälöiden virhevarianssit eivät olleet homogeenisia, mistä johtuen yhtälöt ovat lievästi harhaisia puun koon suhteen. Puun pituuden suhteen harhattomat jäännösvarianssit laskettiin kaavalla (52).

Mikä tahansa regressioyhtälöiden muuttujista voidaan korvata mitatulla arvolla, jota tällöin pidetään eksogeenisena muuttujana. Kyseisen muuttujan määrittävä regressioyhtälö korvataan mitatulla arvolla. Jos läpimitan mittauskorkeus on osakorkeusläpimittojen välillä, käytetään interpolointia. Interpolointikaava sisällytetään uudeksi yhtälöksi malliin. Koska endogeenisten muuttujien ja yhtälöiden lukumäärien tulee olla yhtä suuret, yksi osakorkeusläpimittojen regressioyhtälöistä, yleensä mitattua läpimittaa lähinnä oleva, on jätettävä pois yhtälöryhmästä.

Aiemmissa tutkimuksissa esitettyjen interpolointikaavojen sekä hermiittisen ja luonnollisen kuutiosplini-interpoloinnin sopivuus testattiin. Erot olivat merkityksettömiä. Kuutio-splini valittiin interpolointimenetelmäksi. Jotta interpolointi antaisi harhattomia tuloksia myös rungon tyvessä ja latvassa, lisättiin tyven alapuolelle ja latvan yläpuolelle kuvitellut ylimääräiset läpimitat (kaavat 53 ja 54).

Kun puun pituus oletetaan mitatuksi, yhtälöryhmä linearisoituu, ja se voidaan ratkaista tavanomaisin matriisioperaatioin. Ratkaisu antaa harhattomat estimaatit läpimittojen logaritmeille. Harhattomat läpimittojen estimaatit saadaan kaavalla (56).

Jos läpimitat on mitattu muilta kuin kiinnitetyiltä suhteellisilta osakorkeuksilta, yhtälöryhmään tulee yhtä monta interpolointikaavaa kuin on mitattuja läpimittoja. Hermiittistä tai kuutiosplini-interpolointikaavaa ei voi linearisoida. Siksi on käytettävä iteratiivista ratkaisualgoritmia, esim. kuvassa 2 esitettyä. Mitattua läpimittaa lähimmän kiinteän osakorkeusläpimitan regressioyhtälö korvataan mitatun läpimitan logaritmin ilmaisevalla kaavalla (55), ja yhtälöryhmä ratkaistaan. Mitattua läpimittaa lähinnä olevien kahden yhtälön keskivirheet estimoidaan kaavalla (59). Virhevariansseista aiheutuvat korjaukset tehdään ennen endogeenisten muuttujien numerusten ottoa (kaava 56). Interpolointikaavalla estimoidaan uudet arvot mitatuille läpimitoille. Mitattuja läpimittoja lähinnä olevia perusläpimittoja korjataan ja iterointia jatketaan, kunnes estimoidut läpimitat ovat kyllin lähellä mitattuja.

Liitteessä on esitetty mallilla I lasketut numeeriset runkokäyrät ja osakorkeusläpimittojen keskivirheet (kaava 60) erikokoisille puille. Runkokäyrää voidaan käyttää myös ylemmän läpimitan estimoinnissa. Kuvassa 5 on esitetty mallilla I laskettu kapeneminen d:n ja h:n funktiona. Latvussuhde vaikuttaa voimakkaasti runkomuotoon. Kuvassa 6 on laskettu esimerkkipuun runkokäyrät mallilla II latvussuhteen eri arvoilla.

Tilavuudet integroitiin Huberin kaavalla pätkissä, joiden pituus oli 1/100 puun pituudesta. Integrointi aloitettiin 1 %:n korkeudelta. Läpimitan neliöitä korjattiin ennen integrointia kaavan (61) mukaisesti. Osakorkeusläpimittojen välisten läpimittojen keskivirheet interpoloitiin lineaarisesti. Mallin I antamat tilavuusestimaatit eri kokoisille puille on esitetty taulukossa 9. Taulukossa 10 on esitetty saman mallin antamat tilavuusestimaatit, kun kapeneminen on 3 cm. Taulukossa 11 on esitetty mallin II antamia tilavuusestimaatteja latvussuhteen eri arvoilla.

Puun pituuden mukaan vaihtuvien keskivirheiden käyttö (yhtälö 52) pienentää hieman suurten ja suurentaa pienten puiden tilavuusestimaatteja. Taulukossa 12 on esitetty uudet tilavuusestimaatit (vrt. taulukko 9).

Esimerkkejä mallilla I lasketuista puutavaralajiprosenteista ja rungon arvoista on taulukoissa 13, 14 ja 15. Kuvassa 7 on esitetty tukkipuuprosentti rinnankorkeusläpimitan funktiona latvussuhteen eri arvoilla.

Puun läpimitta-, pituus- ja tilavuuskasvua voidaan estimoida simultaanisen moniyhtälömallin avulla. Kasvun estimointiin voidaan käyttää Jacobin matriisia (kaava 62), joka sisältää regressioyhtälöiden ensimmäiset derivaatat kaikkien endogeenisten muuttujien suhteen. Jacobin matriisia käyttäen ratkaistaan suhteellisten osakorkeuksien läpimittojen ja pituuden kasvut (kaava 63). Jacobin matriisin ratkaisu ei kerro suoraan osakorkeusläpimittojen kasvuja, jos puun pituus samalla kasvaa ja suhteelliset osakorkeudet muuttuvat. Ratkaisun perusteella lasketaan uusi runkokäyrä ja näin saadaan osakorkeusläpimittojen ja tilavuuden muutosestimaatit.

Jacobin matriisi sopii käytettäväksi vain pienillä kasvun arvoilla. Suurilla kasvuilla on sovellettava normaalia erotusmenetelmää.

Runkotilavuuden keskivirheen estimoimiseksi kehitettiin kaava (64). Taulukoissa 19 ja 20 on esitetty näitä keskivirheitä rinnankorkeusläpimitan ja pituuden funktiona. Taulukossa 19 on regressioyhtälöiden (1) . . . (24) keskivirheet oletettu vakioiksi, taulukossa 20 ne on laskettu kaavalla (52).

Tukkipuun tilavuusestimaattien keskivirheet voidaan laskea kaavalla (66). Taulukoissa 21 ja 22 on esitetty näitä keskivirheitä, kun regressioyhtälöiden keskivirheet on laskettu kuten taulukoissa 19 ja 20.

Mallien hyvyys testattiin tutkimusaineistolla. Taulukoissa 23 (koko aineisto) ja 24 (h>8.6 m) on esitetty osakorkeusläpimitta- ja tilavuusestimaattien absoluuttiset ja suhteelliset keskivirheet ja harhat. Latvussuhde vähentää keskivirhettä ja harhaa yli 50 %:n korkeudella olevissa läpimitoissa. Valemuuttujina olevat metsiköiden numerot vähentävät kaikkien osakorkeuksien läpimittojen ja runkotilavuuden keskivirheitä. Tämä on osoitus metsiköiden välisestä vaihtelusta, mikä puoltaa metsikkötunnusten vaikutuksen perusteellisempaa analyysiä mahdollisissa jatkotutkimuksissa.

Tutkimus osoittaa simultaanisten moniyhtälömallien soveltuvan runkokäyrän estimointiin. Malleihin voidaan liittää endogeenisiksi muuttujiksi mitkä tahansa puusta mitattavat tunnukset. Näitä voivat olla esim. latvusta ja kuorta kuvaavat tunnukset. Tällaiset uudet mallit samoin kuin mallit uusille puulajeille saattavat kuitenkin vaatia epälineaarisia ratkaisualgoritmeja.

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
2	2	53	45	43	42	40	36	32	27	22	18	15	15
		7	6	5	5	4	4	3	2	1	1	2	3
2	4	31	26	24	24	22	20	19	16	14	12	10	8
		2	2	1	1	1	0	1	1	1	2	2	2
4	2	102	89	83	82	77	70	63	52	43	35	29	27
		14	11	10	9	8	7	6	4	2	2	3	6
4	4	62	53	49	47	45	41	37	32	27	22	17	13
		5	3	3	2	2	0	1	2	2	3	3	3
4	6	57	48	45	43	41	38	34	31	27	23	18	12
		4	3	2	1	0	1	2	2	2	3	3	3
4	8	54	46	43	41	39	36	33	30	27	23	18	12
		4	2	1	0	1	1	2	2	3	3	3	3
6	2	151	133	123	120	113	103	93	78	65	53	42	38
		20	17	14	13	12	10	8	6	3	3	5	8
6	4	93	81	74	71	67	61	55	48	40	32	24	18
		8	5	4	3	3	1	2	3	3	4	4	5
6	6	85	74	68	64	61	56	51	45	39	32	24	16
		6	4	3	2	1	2	2	3	4	4	5	4
6	8	82	70	64	61	58	54	50	45	40	33	25	16
		6	3	2	0	2	2	2	3	4	4	5	4
6	10	79	67	62	59	56	53	49	45	40	34	26	17
		5	3	2	1	2	2	3	3	4	5	5	4
6	12	77	65	60	58	55	52	48	45	40	35	27	17
		5	3	0	2	2	2	3	3	4	5	5	5
8	2	200	176	162	158	149	135	122	103	86	70	55	49
		26	22	19	18	16	14	11	7	4	4	6	10
8	4	125	109	100	95	90	82	74	63	53	42	31	22
		10	7	5	4	4	1	3	3	4	5	6	6
8	6	114	99	90	85	81	74	68	60	52	42	31	20
		9	6	4	3	1	3	3	4	5	6	6	5
8	8	109	94	86	81	77	71	66	59	52	43	32	20
		8	5	3	1	2	3	3	4	5	6	6	5
8	10	106	90	83	78	75	70	64	58	52	44	33	20
		7	4	3	2	3	3	3	4	5	6	6	5
8	12	103	87	81	77	74	69	64	59	53	45	34	21
		6	3	1	3	3	3	4	4	5	6	6	6
8	14	102	86	79	76	73	69	64	59	53	46	35	22
		6	3	1	3	3	3	4	4	5	6	7	6
8	16	101	84	79	76	73	69	64	59	54	47	37	23
		6	3	2	3	3	3	4	4	5	6	7	6

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
10	2	247	219	200	195	184	167	151	128	107	87	69	59
		33	27	24	22	20	17	14	9	5	5	8	13
10	4	156	137	125	119	112	102	92	79	66	51	38	25
		13	9	6	5	4	1	3	4	5	6	7	7
10	6	143	124	113	107	102	93	85	74	63	51	37	23
		11	7	5	4	1	4	4	5	6	7	7	6
10	8	137	118	108	102	97	89	82	73	64	52	38	23
		10	6	4	1	3	3	4	5	6	7	7	6
10	10	132	113	104	98	93	87	80	72	64	53	39	24
		9	5	3	2	3	4	4	5	6	7	7	6
10	12	129	110	101	96	92	86	79	72	64	54	41	24
		8	4	1	3	3	4	4	5	6	7	8	6
10	14	128	107	99	95	91	85	79	73	65	55	42	25
		8	4	1	3	4	4	4	5	6	8	8	7
10	16	126	106	98	94	90	85	79	73	66	57	44	26
		7	4	2	3	4	4	5	6	6	8	8	7
10	18	125	104	97	94	90	85	79	74	67	58	45	28
		7	3	3	4	4	4	5	6	7	8	9	7
10	20	124	103	97	93	90	85	80	74	68	59	47	29
		7	2	3	4	4	5	5	6	7	8	9	8
12	4	188	166	150	143	135	123	111	94	78	60	44	29
		15	11	8	6	5	1	4	5	6	8	8	8
12	6	172	150	136	128	122	111	101	88	75	60	43	26
		13	9	6	4	1	4	5	6	7	8	8	7
12	8	165	143	129	122	116	106	98	87	75	61	44	26
		12	7	4	1	3	4	5	6	7	8	8	7
12	10	159	136	124	117	112	103	95	86	75	62	46	27
		10	6	4	3	4	5	5	6	7	8	9	7
12	12	156	133	121	115	110	102	95	86	76	63	47	28
		10	5	1	4	4	5	5	6	7	9	9	7
12	14	153	130	119	113	108	101	94	86	77	65	49	29
		9	5	1	4	4	5	5	6	8	9	9	8
12	16	151	127	118	112	108	101	94	87	78	66	51	30
		9	4	3	4	5	5	5	7	8	9	10	8
12	18	150	125	116	112	107	101	94	87	79	68	52	31
		8	3	3	4	5	5	6	7	8	9	10	8
12	20	149	123	115	111	107	101	94	88	80	69	54	33
		8	3	4	5	5	5	6	7	8	10	10	9
12	22	147	122	115	111	107	102	95	89	81	71	56	34
		8	1	4	5	5	6	6	7	8	10	11	9
12	24	147	121	115	112	108	102	96	90	82	72	58	36
		8	1	4	5	5	6	6	7	8	10	11	10

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
14	4	220	195	176	166	158	143	129	109	90	70	50	32
		18	13	9	7	6	1	5	6	7	9	9	8
14	6	201	176	159	150	142	129	118	102	87	68	49	29
		15	10	7	5	2	5	5	7	8	9	9	8
14	8	193	167	151	142	135	124	114	101	87	69	50	29
		14	8	5	1	4	5	5	7	8	9	9	8
14	10	186	160	145	137	130	120	111	99	87	71	52	29
		12	7	4	3	4	5	6	7	8	10	10	8
14	12	182	155	141	134	128	118	110	99	88	72	53	30
		11	6	1	4	5	6	6	7	9	10	10	8
14	14	179	152	139	132	126	117	109	100	89	74	55	32
		11	6	1	5	5	6	6	7	9	10	10	8
14	16	176	149	137	130	125	117	109	100	89	76	57	33
		10	5	3	5	5	6	6	8	9	10	11	9
14	18	174	146	135	130	125	117	109	101	90	77	59	35
		10	4	4	5	5	6	6	8	9	11	11	9
14	20	173	144	134	129	124	117	109	101	91	79	61	36
		10	3	4	5	6	6	7	8	9	11	11	10
14	22	172	142	134	129	124	118	110	102	93	81	63	38
		9	1	5	6	6	7	7	8	9	11	12	10
14	24	171	141	133	129	125	118	111	103	94	82	65	40
		9	1	5	6	6	7	7	8	10	11	12	11
14	26	171	140	133	129	125	119	111	104	95	84	67	42
		8	0	4	6	6	7	7	8	10	12	13	11
14	28	170	139	133	129	125	119	112	105	96	85	69	44
		8	1	5	6	7	7	7	9	10	12	13	12
16	4	252	223	201	190	181	163	147	125	103	78	56	35
		20	14	10	8	7	2	5	7	8	10	10	9
16	6	230	202	182	171	163	147	134	116	98	77	55	32
		17	12	8	6	2	6	6	8	9	10	10	8
16	8	220	192	173	162	155	141	130	114	98	78	56	32
		15	10	6	1	5	5	6	8	9	10	10	8
16	10	212	183	166	156	149	137	126	113	98	79	57	32
		14	8	5	4	5	6	7	8	10	11	11	9
16	12	208	178	162	153	146	135	125	113	99	81	59	33
		13	7	1	5	5	6	7	8	10	11	11	9
16	14	204	174	158	150	144	134	124	113	100	83	61	35
		12	6	1	5	6	7	7	8	10	11	12	9
16	16	201	170	156	149	143	133	124	113	101	85	63	36
		12	6	4	5	6	7	7	9	10	12	12	10
16	18	199	167	154	147	142	133	124	114	102	86	66	38
		11	5	4	6	6	7	7	9	10	12	12	10

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
16	20	198	165	153	147	141	133	124	114	103	88	68	39
		11	4	5	6	7	7	8	9	10	12	13	11
16	22	196	163	152	147	141	133	125	116	104	90	70	41
		10	2	5	7	7	8	8	9	11	13	13	11
16	24	195	161	152	147	141	134	125	117	106	92	72	43
		10	1	5	7	7	8	8	9	11	13	14	12
16	26	195	160	151	147	142	135	126	118	107	94	74	45
		9	0	5	6	7	8	8	9	11	13	14	12
16	28	194	159	151	147	142	135	127	119	108	95	76	47
		9	1	6	7	8	8	8	10	11	13	14	13
16	30	193	157	150	147	142	136	127	119	109	97	78	49
		9	1	6	7	8	8	9	10	11	14	15	13
18	6	259	228	204	192	183	165	151	130	110	85	60	35
		19	13	9	7	2	6	7	9	10	11	11	9
18	8	248	217	195	183	174	158	145	128	110	87	62	34
		17	11	7	1	5	6	7	9	10	12	11	9
18	10	239	207	187	175	167	153	141	126	109	88	63	35
		16	9	6	4	6	7	8	9	11	12	12	9
18	12	234	201	182	171	164	151	140	126	110	90	65	36
		15	8	1	6	6	7	8	9	11	12	12	10
18	14	230	196	178	169	161	150	139	126	111	92	68	37
		14	7	1	6	6	7	8	9	11	13	13	10
18	16	226	192	175	167	160	149	138	126	112	94	70	39
		13	7	4	6	7	8	8	10	11	13	13	10
18	18	224	189	173	165	159	148	138	127	113	95	72	41
		12	5	5	6	7	8	8	10	11	13	13	11
18	20	222	186	172	164	158	148	139	128	114	97	74	43
		12	4	5	7	7	8	9	10	11	13	14	11
18	22	220	183	171	164	158	149	139	129	116	100	77	45
		12	2	6	7	8	9	9	10	12	14	14	12
18	24	219	182	170	164	158	150	140	130	117	102	79	47
		11	1	6	7	8	9	9	10	12	14	15	13
18	26	219	180	170	164	158	150	141	131	118	103	81	49
		10	0	6	7	8	9	9	10	12	14	15	13
18	28	218	178	169	164	159	151	141	132	120	105	83	51
		11	1	6	8	8	9	9	11	12	15	16	14
18	30	217	177	168	164	159	151	142	133	121	107	86	53
		11	2	7	8	9	9	10	11	12	15	16	14
20	6	288	254	227	214	203	183	167	144	121	93	65	37
		22	15	10	7	2	7	8	9	11	12	12	10
20	8	276	242	216	203	193	176	161	142	121	95	67	37
		19	12	7	2	6	7	8	10	11	13	12	10

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
20	10	266	231	207	194	186	170	157	139	121	96	69	37
		17	10	6	5	6	7	8	10	12	13	13	10
20	12	260	224	202	190	182	167	155	139	122	98	71	39
		16	9	2	6	7	8	9	10	12	14	13	10
20	14	255	218	198	187	179	166	154	139	122	100	73	40
		15	8	2	7	7	8	9	10	12	14	14	11
20	16	251	213	195	184	177	165	153	139	123	102	76	42
		15	7	5	7	7	8	9	11	12	14	14	11
20	18	249	210	192	183	176	164	153	140	125	104	78	44
		14	6	5	7	8	8	9	11	12	14	15	12
20	20	246	207	190	182	175	164	153	141	126	106	81	46
		14	5	6	7	8	9	9	11	13	15	15	12
20	22	244	204	189	182	175	165	154	142	127	109	83	48
		13	2	7	8	9	9	10	11	13	15	16	13
20	24	243	202	189	182	175	165	154	143	129	111	86	50
		12	1	7	8	9	10	10	12	13	15	16	13
20	26	242	200	188	181	175	166	155	144	130	113	88	52
		11	0	6	8	9	9	10	12	13	16	17	14
20	28	241	198	187	181	175	166	156	145	131	115	90	55
		12	1	7	9	9	10	10	12	13	16	17	15
20	30	240	196	186	181	175	167	156	146	132	117	93	57
		12	2	7	9	10	10	11	12	14	16	18	15
22	6	317	281	250	235	224	201	183	158	132	101	71	40
		24	16	11	8	3	8	9	10	12	13	13	10
22	8	304	267	238	223	212	193	177	155	132	103	72	39
		21	13	8	2	6	7	8	11	12	14	13	10
22	10	293	254	228	214	204	186	172	152	132	104	74	40
		19	11	7	5	7	8	9	11	13	14	14	11
22	12	286	247	222	209	200	184	170	152	133	107	77	41
		18	10	2	7	7	9	9	11	13	15	14	11
22	14	281	241	218	205	196	182	168	152	134	109	79	43
		17	9	2	7	8	9	10	11	13	15	15	11
22	16	277	235	214	202	194	180	168	152	135	111	82	45
		16	8	5	7	8	9	10	12	13	15	15	12
22	18	273	231	211	201	193	180	167	153	136	113	84	47
		15	6	6	8	8	9	10	12	13	16	16	12
22	20	271	227	209	199	192	179	167	153	137	115	87	49
		15	5	7	8	9	10	10	12	14	16	16	13
22	22	269	224	208	199	192	180	168	155	139	118	90	51
		14	2	7	9	10	10	11	12	14	16	17	14
22	24	267	222	207	199	192	181	169	156	140	120	92	53
		14	2	7	9	10	10	11	13	14	17	17	14

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
22	26	266	220	206	199	192	181	169	157	141	122	95	56
		13	0	7	9	9	10	11	13	14	17	18	15
22	28	265	218	205	198	192	182	170	158	143	124	97	58
		13	%z8	10	10	11	11	13	15	17	18	16	
22	30	264	216	204	198	192	182	171	159	144	126	100	61
		13	2	8	10	11	11	12	13	15	18	19	16
24	8	332	292	260	244	232	210	193	169	143	111	78	42
		23	15	9	2	7	8	9	11	13	15	14	11
24	10	319	278	249	233	222	203	187	166	142	112	79	42
		21	12	8	6	8	9	10	12	14	15	15	11
24	12	313	270	243	228	218	200	185	165	144	115	82	44
		19	11	2	7	8	9	10	12	14	16	16	12
24	14	307	263	237	223	214	197	183	165	145	117	85	45
		18	10	2	8	8	10	10	12	14	16	16	12
24	16	302	257	233	220	211	196	182	165	146	120	87	47
		17	9	6	8	9	10	11	12	14	16	16	13
24	18	298	252	230	218	209	195	182	166	147	122	90	49
		16	7	6	8	9	10	11	13	14	17	17	13
24	20	295	248	228	217	208	195	182	166	148	124	93	51
		16	6	7	9	10	11	11	13	15	17	17	14
24	22	293	245	226	216	208	195	182	168	150	127	96	54
		16	2	8	10	10	11	12	13	15	18	18	14
24	24	291	242	225	216	208	196	183	169	151	129	99	56
		15	2	8	10	11	11	12	14	15	18	19	15
24	26	290	240	224	216	208	196	184	170	153	131	101	59
		14	0	7	9	10	11	12	14	15	18	19	16
24	28	288	238	223	216	208	197	184	171	154	134	104	61
		14	2	8	10	11	12	12	14	16	19	20	16
24	30	287	235	222	215	208	197	185	172	155	136	107	64
		14	2	9	11	11	12	13	14	16	19	20	17
26	8	360	317	282	264	251	227	208	182	154	119	83	44
		25	16	10	2	7	9	10	12	14	16	15	12
26	10	346	302	270	252	241	219	202	179	153	121	85	45
		23	13	8	6	8	10	11	13	15	17	16	12
26	12	339	293	263	246	235	216	199	178	155	123	88	46
		21	12	2	8	9	10	11	13	15	17	17	12
26	14	332	285	257	242	231	213	198	178	156	126	90	48
		20	10	2	8	9	10	11	13	15	17	17	13
26	16	327	279	252	238	228	211	196	178	156	128	93	50
		19	9	6	9	10	11	11	13	15	18	18	13
26	18	322	273	249	236	226	210	196	178	158	130	96	52
		18	8	7	9	10	11	12	14	15	18	18	14

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
26	20	319	269	246	234	225	210	196	179	159	133	99	54
		18	6	8	10	10	11	12	14	16	18	19	14
26	22	317	266	245	234	225	211	197	180	161	135	102	57
		17	3	9	10	11	12	13	13	16	19	19	15
26	24	315	263	243	233	225	211	197	181	162	138	105	59
		16	2	9	11	11	12	13	15	16	19	20	16
26	26	313	260	242	233	224	212	198	183	164	140	108	62
		15	0	8	10	11	12	13	15	16	19	20	17
26	28	312	257	241	232	224	212	199	184	165	143	111	65
		15	2	9	11	12	13	13	15	17	20	21	17
26	30	310	255	239	232	224	213	199	185	166	145	113	67
		15	2	10	12	12	13	13	15	17	20	22	18
28	8	388	342	304	284	270	244	224	195	165	127	88	47
		27	17	10	2	8	9	11	13	15	17	16	12
28	10	373	326	290	271	259	236	217	192	164	129	90	47
		25	15	9	7	9	10	12	14	16	18	17	12
28	12	365	316	283	265	253	232	214	191	165	131	93	48
		23	13	2	9	9	11	12	14	16	18	18	13
28	14	358	308	277	260	249	229	212	191	166	134	96	50
		22	11	2	9	10	11	12	14	16	18	18	13
28	16	352	301	271	256	245	227	211	190	167	136	99	52
		20	10	6	9	10	11	12	14	16	19	19	14
28	18	347	295	268	253	243	226	210	191	169	139	102	55
		19	8	7	10	11	12	12	15	17	19	19	15
28	20	344	290	265	252	242	226	210	192	170	141	105	57
		19	7	8	10	11	12	13	15	17	20	20	15
28	22	341	286	263	251	241	226	211	193	172	144	108	60
		18	3	9	11	12	13	14	15	17	20	20	16
28	24	339	283	261	250	241	226	211	194	173	147	111	62
		17	2	9	11	12	13	14	16	18	20	21	17
28	26	337	280	260	250	241	227	212	195	175	149	114	65
		16	0	8	11	12	13	13	16	18	21	22	17
28	28	335	277	258	249	241	227	213	196	176	152	117	68
		16	2	10	12	13	14	14	16	18	21	22	18
28	30	333	274	257	249	240	228	213	197	178	154	120	71
		16	2	10	12	13	14	14	16	18	22	23	19
30	10	400	350	311	291	277	252	232	204	175	136	95	49
		26	16	10	7	9	11	12	15	17	19	18	13
30	12	391	340	304	284	271	248	229	204	176	139	98	51
		24	13	2	9	10	12	13	15	17	19	19	14
30	14	383	330	297	278	266	245	227	203	177	142	101	53
		23	12	2	10	10	12	13	15	17	20	19	14

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
30	16	377	322	291	274	262	242	225	203	178	145	104	55
		22	11	7	10	11	12	13	15	17	20	20	15
30	18	372	316	286	271	260	241	224	203	179	147	107	57
		21	9	8	10	11	12	13	15	18	20	20	15
30	20	368	311	283	269	258	241	224	204	181	150	111	60
		20	7	9	11	12	13	14	16	18	21	21	16
30	22	365	307	281	268	258	241	225	205	183	153	114	62
		19	3	10	12	13	14	14	16	18	21	22	17
30	24	363	303	279	267	257	241	225	207	184	156	117	65
		19	2	10	12	13	14	15	17	19	22	22	17
30	26	361	300	278	267	257	242	226	208	186	158	120	68
		17	0	9	12	13	14	14	17	19	22	23	18
30	28	359	297	276	266	257	242	227	209	187	161	124	71
		17	2	10	13	14	14	15	17	19	22	23	19
30	30	357	294	274	266	256	243	227	210	189	163	127	74
		17	3	11	13	14	15	15	17	19	23	24	20
32	10	427	374	332	310	295	268	247	217	186	144	100	51
		28	17	10	8	10	12	13	16	18	20	19	14
32	12	418	363	324	302	289	264	244	217	187	147	103	53
		26	14	3	10	11	12	13	16	18	20	20	14
32	14	409	353	316	296	284	261	241	216	188	150	107	55
		25	13	2	10	11	13	14	16	18	21	20	15
32	16	402	344	310	291	279	258	239	216	189	153	110	57
		23	12	7	11	12	13	14	16	18	21	21	15
32	18	396	337	305	288	277	256	238	216	190	155	113	60
		22	9	8	11	12	13	14	16	19	21	21	16
32	20	392	332	302	286	275	256	238	217	192	158	116	62
		22	8	9	12	13	14	15	17	19	22	22	17
32	22	389	327	299	285	274	256	239	218	193	161	120	65
		21	3	10	13	14	15	15	17	20	22	23	17
32	24	387	324	297	284	274	256	239	219	195	164	123	68
		20	2	10	13	14	15	15	18	20	23	23	18
32	26	384	320	295	283	273	257	240	220	197	167	127	71
		18	0	10	12	14	15	15	18	20	23	24	19
32	28	382	317	294	283	273	257	240	222	198	169	130	74
		19	2	11	14	15	15	16	18	20	24	25	20
32	30	380	313	292	282	272	258	241	223	199	172	133	77
		19	3	12	14	15	16	16	19	21	24	25	21
34	10	454	398	353	329	314	285	262	230	196	152	105	53
		30	18	11	8	11	12	14	17	19	21	20	14
34	12	444	386	344	321	307	280	258	229	198	155	109	55
		28	15	3	10	11	13	14	17	19	21	21	15

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height – *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
34	14	434	375	336	314	301	276	256	229	199	158	112	57
		26	14	3	11	12	13	14	17	19	22	21	15
34	16	427	366	329	309	296	273	253	228	199	161	115	59
		25	12	8	11	12	14	15	17	20	22	22	16
34	18	421	359	324	306	293	272	252	228	201	164	119	62
		23	10	9	12	13	14	15	17	20	22	22	16
34	20	417	353	320	303	292	271	252	229	202	167	122	65
		23	8	10	12	14	15	16	18	20	23	23	17
34	22	413	348	318	302	290	271	253	230	204	170	126	68
		22	3	11	13	15	16	16	18	21	24	24	18
34	24	410	344	315	301	290	271	253	232	206	173	129	71
		21	3	11	14	15	16	16	19	21	24	24	19
34	26	408	340	313	300	289	272	254	233	207	176	133	74
		19	0	10	13	14	15	16	19	21	24	25	20
34	28	405	336	311	300	289	272	254	234	209	178	136	77
		20	2	12	14	15	16	17	19	21	25	26	21
34	30	403	333	309	299	288	272	255	235	210	181	139	80
		20	3	12	15	16	17	17	20	22	25	26	21
36	12	470	410	364	339	324	296	273	242	208	163	114	57
		29	16	3	11	12	14	15	18	20	22	21	15
36	14	460	398	356	333	318	292	270	241	209	166	117	59
		28	15	3	12	13	14	15	18	20	23	22	16
36	16	452	388	348	327	313	288	268	241	210	169	121	62
		26	13	8	12	13	15	16	18	21	23	23	16
36	18	445	380	343	323	310	287	266	241	211	172	124	64
		25	11	10	12	13	15	16	18	21	24	23	17
36	20	441	374	339	321	308	286	266	242	213	175	128	67
		24	9	11	13	14	15	16	19	21	24	24	18
36	22	437	369	336	319	307	286	266	243	215	178	131	70
		23	4	12	14	15	16	17	19	22	25	25	19
36	24	434	364	333	318	306	286	267	244	216	181	135	73
		22	3	12	14	16	17	17	20	22	25	26	20
36	26	431	360	331	317	305	287	268	245	218	184	139	76
		20	0	11	14	15	16	17	20	22	26	26	20
36	28	429	356	329	316	305	287	268	246	220	187	142	80
		21	2	12	15	16	17	18	20	22	26	27	21
36	30	426	352	327	315	304	287	269	247	221	189	145	83
		21	3	13	16	17	18	18	21	23	27	28	22
38	12	497	433	385	358	342	312	288	255	219	171	119	59
		31	17	3	12	13	15	16	19	21	24	22	16
38	14	486	420	375	351	336	307	285	254	220	174	122	62
		29	15	3	12	13	15	16	19	21	24	23	16

Appendix. Relative-height diameters and their standard errors (mm).
Liite 1. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$	h	relative height – *suhteellinen korkeus*											
cm	m	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
38	16	477	410	367	344	330	304	282	253	220	177	126	64
		28	14	9	13	14	15	16	19	22	24	24	17
38	18	470	401	361	340	326	302	280	253	222	180	129	67
		26	11	10	13	14	16	17	19	22	25	24	18
38	20	465	395	357	338	324	301	280	254	223	183	133	70
		26	9	11	14	15	16	17	20	22	25	25	19
38	22	461	389	354	336	323	301	280	255	225	186	137	73
		25	4	12	15	16	17	18	20	23	26	26	19
38	24	458	385	351	335	322	301	281	256	227	189	141	76
		24	3	12	15	16	17	18	21	23	26	27	20
38	26	455	380	349	334	321	301	281	258	229	193	145	79
		21	0	11	15	16	17	18	21	23	27	27	21
38	28	452	376	346	333	321	302	282	259	230	195	148	82
		22	3	13	16	17	18	19	21	24	27	28	22
38	30	449	371	344	332	320	302	282	260	232	198	152	86
		22	3	14	17	18	18	19	22	24	28	29	23
40	12	523	456	405	377	360	328	302	267	229	179	124	61
		33	18	3	12	13	15	17	20	22	25	23	16
40	14	511	443	395	369	353	323	299	266	230	182	127	64
		31	16	3	13	14	16	17	20	23	25	24	17
40	16	502	432	386	362	347	319	296	265	231	185	131	66
		29	15	9	13	15	16	17	20	23	25	25	18
40	18	494	423	380	358	343	317	294	265	232	188	135	69
		27	12	11	13	15	16	17	20	23	26	25	18
40	20	489	416	376	355	341	316	294	266	234	191	139	72
		27	10	12	15	16	17	18	21	23	26	26	19
40	22	486	410	372	353	339	316	294	267	236	194	143	75
		26	4	13	16	17	18	19	21	24	27	27	20
40	24	482	405	369	352	338	316	295	269	238	198	147	78
		25	3	13	16	17	18	19	22	24	28	28	21
40	26	478	400	366	350	337	316	295	270	239	201	150	82
		23	0	12	15	17	18	19	22	24	28	28	22
40	28	475	395	364	349	337	316	296	271	241	204	154	85
		23	3	14	17	18	19	20	22	25	29	29	23
40	30	472	391	361	348	336	317	296	272	242	207	158	89
		23	4	14	17	18	19	20	23	25	29	30	24
42	14	537	466	415	387	370	339	313	279	241	190	132	66
		32	17	3	14	15	17	18	21	24	26	25	18
42	16	527	454	405	380	364	334	310	278	241	193	136	68
		31	15	10	14	15	17	18	21	24	26	26	18
42	18	519	444	398	375	360	332	308	278	243	196	140	71
		29	12	11	14	16	17	18	21	24	27	26	19

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$	h	relative height – *suhteellinen korkeus*											
cm	m	.01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
42	20	513	437	394	372	357	331	308	279	244	199	144	74
		28	10	12	15	17	18	19	22	24	27	27	20
42	22	510	431	390	370	355	330	308	280	246	203	148	77
		27	4	14	16	18	19	20	22	25	28	28	21
42	24	506	425	387	368	354	331	308	281	248	206	152	81
		26	3	14	17	18	19	20	23	25	29	29	22
42	26	502	420	384	367	353	331	309	282	250	209	156	84
		24	0	12	16	18	19	20	23	25	29	29	23
42	28	498	415	381	366	353	331	309	283	252	212	160	88
		24	3	14	18	19	20	21	23	26	30	30	24
42	30	495	410	379	365	352	331	309	284	253	215	164	91
		24	4	15	18	19	20	21	24	26	30	31	25
44	14	562	488	434	405	388	354	328	291	251	197	137	68
		34	18	3	14	15	17	18	22	25	27	26	18
44	16	552	476	424	397	380	349	324	290	252	200	141	71
		32	16	10	15	16	18	19	22	25	28	27	19
44	18	543	465	417	392	376	347	322	290	253	204	145	73
		30	13	12	15	16	18	19	22	25	28	27	20
44	20	538	458	412	389	374	346	322	291	255	207	149	77
		30	11	13	16	17	19	20	23	25	29	28	20
44	22	534	452	408	387	371	345	322	292	257	211	154	80
		28	5	14	17	19	20	21	23	26	29	29	21
44	24	529	446	405	385	370	345	322	293	259	214	158	83
		27	3	14	17	19	20	21	24	26	30	30	22
44	26	525	440	402	384	369	345	322	294	260	217	162	87
		25	0	13	17	18	20	20	24	26	30	31	23
44	28	521	435	399	382	368	346	323	295	262	221	166	90
		25	3	15	18	20	21	21	24	27	31	31	24
44	30	518	429	396	381	367	346	323	296	264	224	170	94
		25	4	16	19	20	21	22	25	27	31	32	25
46	14	588	511	454	423	405	370	342	304	261	205	142	70
		35	19	3	15	16	18	19	23	26	28	27	19
46	16	577	498	443	415	397	364	338	302	262	208	146	73
		33	17	10	15	17	18	20	23	26	29	27	19
46	18	568	487	436	409	393	362	336	302	263	211	150	76
		31	14	12	15	17	19	20	23	26	29	28	20
46	20	562	479	431	406	390	361	336	303	265	215	155	79
		31	11	13	17	18	20	21	24	26	30	29	21
46	22	558	472	426	403	388	360	335	304	267	219	159	82
		30	5	15	18	19	21	21	24	27	30	30	22
46	24	553	466	423	402	386	360	336	305	269	222	163	86
		28	3	15	18	20	21	22	25	27	31	31	23

Appendix. Relative-height diameters and their standard errors (mm).
Liite. Läpimitat ja niiden keskivirheet (mm) suhteellisilla osakorkeuksilla.

$d_{1.3}$ cm	h m	relative height — *suhteellinen korkeus* .01	.05	.1	.15	.2	.3	.4	.5	.6	.7	.8	.9
46	26	549	460	419	400	385	360	336	306	271	226	168	89
		26	0	14	18	19	20	21	25	27	31	32	24
46	28	545	454	416	399	384	360	336	308	273	229	172	93
		27	3	16	19	20	22	22	25	28	32	33	25
46	30	541	449	413	397	383	360	337	309	274	232	176	97
		27	4	16	20	21	22	23	26	28	33	33	26
48	16	602	520	462	432	414	379	352	314	272	216	151	75
		35	18	11	16	17	19	21	24	27	30	28	20
48	18	592	508	454	427	409	377	350	314	273	219	156	78
		33	14	13	16	18	19	21	24	27	30	29	21
48	20	586	500	449	423	406	376	349	315	276	223	160	81
		32	12	14	17	19	20	21	25	27	31	30	22
48	22	582	493	444	420	404	375	349	316	277	227	164	84
		31	5	15	19	20	21	22	25	28	31	31	23
48	24	577	486	440	418	402	375	349	317	279	230	169	88
		30	4	15	19	20	22	23	26	28	32	32	24
48	26	572	480	437	417	401	375	349	318	281	234	173	92
		27	0	14	18	20	21	22	25	28	32	33	25
48	28	568	474	434	415	400	375	350	320	283	237	177	96
		28	3	16	20	21	22	23	26	29	33	34	26
48	30	563	468	430	413	399	375	350	321	284	240	181	99
		28	4	17	21	22	23	24	27	29	34	34	27
50	16	627	542	481	450	431	395	366	327	283	223	156	77
		36	18	11	17	18	20	21	25	28	31	29	20
50	18	617	530	473	444	425	392	364	326	284	227	161	80
		34	15	13	17	18	20	22	25	28	31	30	21
50	20	610	521	467	440	423	391	363	327	286	231	165	83
		33	12	15	18	20	21	22	26	28	32	31	22
50	22	605	514	462	437	420	389	362	328	287	234	170	87
		32	5	16	19	21	22	23	26	29	33	32	23
50	24	600	507	458	435	418	389	363	329	290	238	174	90
		31	4	16	20	21	22	23	27	29	33	33	24
50	26	595	500	454	433	417	389	363	330	292	242	179	94
		28	0	15	19	21	22	23	26	29	34	34	25
50	28	591	494	451	431	416	389	363	332	293	245	183	98
		29	3	17	21	22	23	24	27	30	34	35	26
50	30	586	488	448	430	414	389	363	333	295	248	187	102
		29	4	18	21	23	24	25	28	30	35	36	27

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Taper curve models based upon the simultaneous equations are derived. The data consisted of 492 Scots pines from Southern Finland. Two systems of simultaneous equations were constructed, one without the crown ratio and the other with the crown ratio as an exogenous variable. The endogenous variable consisted of 24 relative-height diameters and the height of the tree. The parameters of the model were derived by the ordinary least squares method.
The models were applied to the derivation of taper curves, stem volumes, timber assortment percentages, and stem values. An experiment was also made to derive diameter and height increments from the taper curve model.
A Fortran program which may utilise up to 20 measured diameters, height, and the crown ratio in the estimation of the taper curve, stem volume, timber assortment percentages, and the standard errors of the stem and saw log volumes is available from the authors.

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