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ON THE CONSTRUCTION OF MONOTONY PRESERVING TAPER CURVES

MONOTONISUUDEN SÄILYTTÄVIEN RUNKO-KÄYRIEN MUODOSTAMISESTA

Aatos Lahtinen

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Seloste

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A monotony preserving taper curve can be constructed by using a quadratic spline. An algorithm is presented which is suitable for this purpose. It is used to the construction of a taper curve when several measured diameters of a tree are available. These taper curves are formed for different sets of measurements and their properties are evaluated. It appears that the monotony preserving quadratic spline can give a better taper curve than the usual cubic spline.

Monotonisuuden säilyttävä runkokäyrä voidaan konstruoida käyttäen neliöllistä splini-funktiota. Tarkoitukseen sopiva algoritmi esitetään ja sitä käytetään runkokäyrän muodostukseen, kun puusta on mitattu useita läpimittoja. Runkokäyriä muodostetaan erilaisia mittaustietoja käyttäen ja saatujen käyrien ominaisuuksia tutkitaan. Osoittautuu, että monotonisuuden säilyttävä neliöllinen splini voi antaa paremman runkokäyrän kuin tavallinen kuutiollinen splini-funktio.

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Korjaus julkaisuun

Matti Leikola. 1987. Suomalaiset metsätieteelliset väitöskirjat ja niiden laatijat. Acta Forestalia Fennica 199.

Liite 1:een (s. 34–35) lisätään **M. Nuorteva,** väittelyvuosi 1956.

Correction to

Matti Leikola. 1987. Academic dissertations and doctors in forestry sciences in Finland. Acta Forestalia Fennica 199.

The name of **M. Nuorteva,** year of disputation 1956, is added to Appendix 1 (p. 34–35), which lists the doctors included in the study.

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PREFACE

This investigation was carried out in the Department of Mathematics, University of Helsinki and funded by the Academy of Finland. The sample tree material was provided by the Finnish Forest Research Institute. The Ministry of Education provided the computer time on a VAX-8600 computer situated in the Finnish State Computing Centre. The computer programs were written, tested and run by Mrs. Kaija Laurila with great care and diligence. I want to express my warm thanks to her and to all others who have helped in the various stages of this investigation.

Helsinki, March 1987

Aatos Lahtinen

1. INTRODUCTION

The taper curve of a tree is a central concept in forest mensuration. The taper curve gives a mathematical model of a tree which can be used as a basis of all the tasks of forest inventory and mensuration. The form of a tree may have a simple apparence but the construction of its mathematical model is a demanding task. In spite of a long line interest and a large number of publications no solution has appeared which satisfies all the relevant conditions.

Lahtinen and Laasasenaho (1979) studied the determination of the taper curve in the case where several diameters measured along the stem were available. They showed that for such situations one can construct an accurate taper curve by using cubic interpolating splines.

A standard tree tapers off monotonically upwards, therefore its taper curve should have the same property. However, the taper curve constructed by using cubic interpolating splines produces a slight oscillation. As a result this taper curve may be unmonotonic for a monotonically tapering tree and may exaggregate the unmonotoniness of a nontapering tree. These phenomena are to be seen from the taper curve CO15 in Figures 2 and 3. This oscillation is an intrisic property of the interpolating cubic spline (cf. De Boor 1978) and no way has been found of dispensing with it.

A quadratic spline is able to interpolate monotonic data monotonically. It can also match the convexity and concavity of the data (Schumaker 1983). For this reason we investigated the suitability of the quadratic spline for a taper curve.

The necessary concepts are introduced in section two. After this a mathematical algorithm is presented which produces an interpolating quadratic spline preserving the monotony of the data. The mathematical results on which the algorithm is based are published separately (Lahtinen 1988).

In section three the algorithm is implemented to construct the taper curve. The criteria are introduced on which we examine the quality of resulting taper curves by using sample trees and determine the free parameters of the taper curves. With the results of determination of parameters we produce taper curves which are analysed in section four.

The suitability of an monotony preserving quadratic spline as a taper curve is affirmed in section five together with observations of the number of measurements needed in the construction.

2. MONOTONY PRESERVING QUADRATIC SPLINES

21. Spline functions

One of the main problems in the approximation theory is the treatment of a function for which only incomplete information is available. The function may be known e.g. only at some discrete points or it may be known to be a solution of an equation which cannot be solved exactly. In such situations the function must be replaced by a known function which agrees with the known facts of the original function sufficiently well and whose properties are suitable for operations in question.

Spline functions form a class of approximating functions which can be adapted for several different purposes. Polynomial splines are easiest to handle. A polynomial spline of degree n consists of polynomial pieces of degree at most n which are joined together at so called breakpoints so that the resulting function has continuous derivatives up to the order n-1. A clear exposition on polynomial splines is to be found in De Boor (1978). Schumaker (1981) provides more information about splines in general.

The most common polynomial spline is the cubic spline. It is a twice continuously differentiable function which consists of polynomial pieces of degree at most three. Lahtinen and Laasasenaho (1979) have given the necessary algorithms for construction of cubic splines as well as the most important properties of them. An interpolating cubic spline has minimal curvature among twice differentiable interpolating functions. It can be uniquely determined by two initial values and it can be evaluated by using numerically stable algorithms. If a function is continuously differentiable then it can be interpolated by a cubic spline with a prescribed accuracy by using a sufficient high number of interpolating points. An interpolating polynomial does not always have this property (De Boor 1978).

Another common polynomial spline is the quadratic spline. It is a continuously differentiable function consisting of polynomial pieces of degree at most two. An interpolating quadratic spline can be uniquely determined by using one initial condition if the interpolating points coincide with the breakpoints. It can be evaluated by very simple algorithms, but they are not as stable as the algorithm for cubical spline (Lahtinen and Laasasenaho 1979). Mettke et al. (1982) have shown that a continuously differentiable function can be interpolated by a quadratic spline with a prescribed accuracy if the number of interpolating points is sufficiently high.

In the construction of a quadratic or cubic spline the derivatives at breakpoints are free parameters at first. They are fixed in the construction so that the spline has the required degree of smoothness. It remains one or two degrees of freedom with which the form of the spline can be affected. The effect of these degrees of freedom is always global, that is a change in their values changes the spline on the whole interval of definition.

22. Monotony preserving splines

In some situations an approximation is wanted not only to the values of the function but also to its shape. The values of the function are known at some discrete set of points but no precise information about the shape is available.

A natural method is to use the shape of the discrete set of function values as an approximation to the shape of the function. This means a construction of an approximating function which interpolates at the known points and which preserves the essential shape of the point set. This essential shape of the point set is in our case piecewise monotony either alone or together with convexity and concavity. In the former case the approximation is said to *preserve the monotony* and in the latter to *preserve the shape*. The mathematical formulation of these terms will be given later on.

The question is how to find an interpolating function which could be able to preserve the monotony or the shape. An interpolating cubic spline does not preserve the monotony despite its many favorable properties. Passow (1974) has shown this with a counterexample. The tests of Lahtinen and Laasasenaho (1979) show that the disturbing of the monotony is quite common in practice. According to De Boor (1978) the main reason for the oscillation is the continuity of the second derivative.

This observation turns the attention to the quadratic spline which has only one continuous derivative. McAllister and Roulier (1978) have shown, however, that the quadratic spline is not always able to preserve the monotony and the convexity or concavity. On the other hand, they established that one can always transform a quadratic spline to a shape preserving one by adding new breakpoints between interpolating points. On this basis they have developed an algorithm which gives a shape preserving quadratic spline (McAllister and Roulier 1981a). The algorithm is available also in the form of a FORTRAN-program (McAllister and Roulier 1981b). An exposition of its ability to create a taper curve will be published later on.

Schumaker (1983) has given another idea how to construct a shape preserving quadratic spline. This seems to lead to a simpler and more flexible method than the one of McAllister and Roulier. Therefore we take the idea of Schumaker as a starting point in the construction of an algorithm for a monotony or shape preserving spline. The mathematical results needed for the algorithm are to be found in Lahtinen (1988). Here we are interested only in the use of the algorithm.

23. Statement of the problem

We start by defining our aim mathematically. For this purpose we need some concepts.

Let [a,b] be an interval on the real axis and $(x_i)_1^n$ a division of the interval so that

$$-\infty < a = x_1 < x_2 < \ldots < x_{n-1} < x_n = b < \infty$$

and let $(y_i)_1^n$, be a given set of real numbers. We use the notation

$$\Delta x_i = x_{i+1} - x_i,\ \Delta y_i = y_{i+1} - y_i,\ i = 1, \ldots, n-1$$

for differences, and for divided differences we use

$$\delta_i = \frac{\Delta y_i}{\Delta x_i},\ i = 1, \ldots, n-1.$$

The point set $D = (x_i, y_i)_1^n$, is called *increasing* (respectively *decreasing)* if the set $(y_i)_1^n$, is increasing (resp. decreasing). The set D is called *convex* (resp. *concave)* if the set of divided differences is increasing (resp. decreasing.)

For motivation of this terminology we consider the situation where there exists a function g so that $g(x_i) = y_i$, $i = 1, \ldots, n$. If g is increasing (resp. decreasing) on the interval, then also D is increasing (resp. decreasing) and if g is convex (resp. concave) on [a,b], then also D has the same property.

Both increasing and decreasing forms are called *monotones.* A *piecewise monotone function* or *set* consists of parts each of which is monotone. A point set which consists of convex or concave parts requires some consideration. Suppose for example that for the point set D we have $\delta_1 < \delta_2 < \delta_3 > \delta_4 > \delta_5$. This means that D is convex on $[x_1, x_4]$ and concave on $[x_3, x_6]$. On the interval $[x_3, x_4]$ D is thus both convex and concave. For clarity we exclude this "interval of inflection" from both convex and concave sets. According to this convention we say that D is *convex* (resp. *concave*) on a subinterval $[x_i, x_j]$ of [a,b], if $(\delta_{i-1}, \ldots, \delta_j)$ is increasing (resp. decreasing).

We say that a function f interpolating at a point set D (so that $s(x_i) = y_i$, $i = 1, \ldots, n$) is *monotony preserving* if it is increasing (resp. decreasing) on the same intervals as D. If it is also convex (resp. concave) on the same intervals as D then it is called *shape preserving*. The curvature of the function f is not considered at intervals of inflection. We are primarily interested in the monotony. Our problem can be formulated as follows:

Suppose that a point set D is given on an interval [a,b]. Construct a quadratic spline s on [a,b] interpolating at the set D so that s is monotony preserving.

The solution of this problem is called a *monotony preserving spline.* If the solution preserves also the convexity and concavity then it is called *a shape preserving spline*. All splines in this article are quadratic splines unless otherwise stated.

24. Spline algorithm

A common interpolating spline has one or two degrees of freedom. This is not sufficient for the construction of the monotony preserving spline we want (McAllister and Roulier 1978). So we must introduce more degrees of freedom.

A natural way to begin is to leave all the derivatives at interpolating points as free parameters. This is not yet sufficient but leads however to the desired solution in many different ways. The system we used was to introduce additional parameters in the form of new breakpoints situated between consecutive interpolating points. We can form explicit conditions under which the spline is monotony or shape preserving. The method is iterative, which means that any parameter can be changed in the process of calculation if its value is not suitable. The mathematical background of this method is to be found in the articles of Schumaker (1983) and Lahtinen (1988). We are only interested in the final form of the algorithm here.

The algorithm:

Initial information:
- the number of interpolating points: n
- the set of interpolating points: $D = ((x_i,y_i))_1^n,\ x_1 < \ldots < x_n$
- parameters for derivatives: $(a_i)_1^n,\ 0 < a_i < \infty,\ i = 1, \ldots, n$
- parameters for breakpoints: $(b_i)_1^{n-1},\ 0 < b_i < 1,\ i=1, \ldots, n-1$

(The meaning of parameters will be explained at a later stage of the algorithm.)

Step 1: Compute auxiliary quantities

$$\Delta x_i = x_{i+1} - x_i,\ \Delta y_i = y_{i+1} - y_i,\ i = 1, \ldots, n-1$$
$$l_i = ((\Delta x_i)^2 + (\Delta y_i)^2)^{1/2},\ \delta_i = \Delta y_i/\Delta x_i,\ i = 1, \ldots, n-1$$
$$\mu_i = \frac{l_{i-1}\,\delta_{i-1} + l_i\,\delta_i}{l_{i-1} + l_i},\ i = 2, \ldots, n-1.$$

Step 2: Compute derivatives at interpolating points.

At a point x_i the derivative m_i of the spline as function of a_i is:

$$m_1 = \delta_1 + a_1\,(\delta_1 + \mu_2)$$

$$m_i = \begin{cases} a_i\mu_i, & \text{if } \delta_i\delta_{i-1} > 0 \\ 0, & \text{if } \delta_i\,\delta_{i-1} \leq 0 \end{cases},\ i = 2, \ldots, n-1$$

$$m_n = \delta_{n-1} + a_n\,(\delta_{n-1} + \mu_{n-1}).$$

These derivatives are not final. In some cases in step 3 it is necessary to change some derivatives in order to get a monotony preserving spline.

Step 3: The expression of the spline is formed separately for each interval $[x_i,x_{i+1}]$. There are several alternatives depending on the derivatives (m_i) and divided differences (δ_i).

Alternative 31: $m_i + m_{i+1} = 2\,\delta_i$.

In this case the spline has in the interval $[x_i, x_{i+1}]$ the expression

$$s_i(x) = y_i + m_i(x-x_i) + \frac{1}{2}\frac{\Delta m_i}{\Delta x_i}(x-x_i)^2.$$

If this is not the case, then we have

Alternative 32: $m_i + m_{i+1} \neq 2\delta_i$.

In this case the spline has on the interval $]x_i, x_{i+1}[$ a breakpoint ξ_i whose location will be specified within certain limits. For these limits we introduce three auxiliary points:

$$u_i = x_i + \Delta x_i\,\frac{m_{i+1} - 2\delta_i}{\Delta m_i}$$

$$v_i = x_i + 2\Delta x_i\,\frac{m_{i+1} - \delta_i}{\Delta m_i}$$

$$w_i = x_{i+1} + 2\Delta x_i\,\frac{m_i - \delta_i}{\Delta m_i}.$$

Now we can present the different cases where the breakpoint ξ_i is chosen as a function of b_i, $0 < b_i < 1$.

Case 321: $(m_{i+1} - \delta_i)(m_i - \delta_i) < 0$

This leads to a shape preserving spline if

$$\xi_i = \begin{cases} x_i + b_i\,(v_i-x_i), & \text{when } |m_{i+1} - \delta_i| < |m_i - \delta_i| \\ w_i + b_i\,(x_{i+1}-w_i), & \text{when } |m_{i+1}-\delta_i| > |m_i-\delta_i|. \end{cases}$$

Notice that in this case always $|m_{i+1}-\delta_i| \neq |m_i-\delta_i|$.

Remark. If the point set D is convex or concave on the interval $[x_i, x_{i+1}]$ then the shape is preserved if the derivative parameters a_i and a_{i+1} are chosen so that we get this case 321. This can always be arranged by choosing a_i and a_{i+1} to be suffiently near the value 1 (cf. Lahtinen 1988). Table 4 contains limits for a_i in a typical situation.

Case 322: $(m_{i+1} - \delta_i)(m_i - \delta_i) \geq 0$

In this case the spline has a point of inflection at the breakpoint ξ_i and cannot be either convex or concave on the whole interval. The choice of ξ_i affects the monotony of the spline as follows:

Case 3221: $m_i \neq m_{i+1}$ and $2|\delta_i| > \min\,(|m_i|, |m_{i+1}|)$

The spline is monotone on $[x_i, x_{i+1}]$ if

$$\xi_i = b_i x_{i+1} + (1 - b_i)\max(x_i,u_i).$$

Case 3222: $m_i = m_{i+1}$ and $2|\delta_i| \geq |m_i|$

The spline is monotone on $[x_i,x_{i+1}]$ if

$$\xi_i = x_i + b_i\,(x_{i+1} - x_i).$$

Case 3223: $2\,|\delta_i| \leq \min(|m_i|,|m_{i+1}|)$, where there is a strict inequality if $m_i = m_{i+1}$.

In this case the spline is unmonotonic on $[x_i,x_{i+1}]$ for all choices of ξ_i. Therefore, if this is the case, the value of the derivatives has to be changed in order to obtain monotony. We replace the value of m_{i+1} by $3/2\ \delta_i$ for simplicity. Because the value of m_i is not altered, the representation of the the spline on the interval $[x_{i-1},x_i]$ needs no recalculation. After the change of derivatives the construction returns again to the beginning of Step 3.

When the situation of the breakpoint ξ_i is fixed, then we can form the expression of the spline on the interval $[x_i,x_{i+1}]$ according to the following formula:

$$s_i(x) = \begin{cases} y_i + m_i(x-x_i) + C_i(x-x_i)^2, & \text{if } x \in [x_i,\xi_i] \\ D_i + E_i(x-\xi_i)+F_i(x-\xi_i)^2, & \text{if } x \in\,]\xi_i,x_{i+1}] \end{cases}$$

The coefficients are

$$C_i = \frac{1}{2}\,\frac{E_i-m_i}{\xi_i-x_i}$$

and

$$D_i = y_i + m_i(\xi_i-x_i) + C_i(\xi_i-x_i)^2,$$

$$E_i = 2\delta_i - \frac{\xi_i-x_i}{\Delta x_i}\,m_i - \frac{x_{i+1}-\xi_i}{\Delta x_i}\,m_{i+1}$$

$$F_i = \frac{1}{2}\,\frac{m_{i+1}-E_i}{x_{i+1}-\xi_i}.$$

Notice that $D_i = s(\xi_i)$ and $E_i = s'(\xi_i)$. Now the spline is formed for the interval $[x_i,x_{i+1}]$ and we can proceed the following intervals until the whole spline is constructed.

This algorithm gives a *local spline*. It means that a change of a parameter doesn't affect the spline on the whole interval of definition but only in some neighbourhood of the point where the change of parameter was made. This locality is an essential feature of the construction which makes it possible to obtain a monotony or shape preserving spline. It also means that we can make local adjustments to a spline. For example suppose that the spline models a function which is monotone on an interval $[x_i,x_{i+1}]$ but our choice of derivatives leads to the case 3223 in the algorithm, which means unmonotony. Then we need only to redefine the derivative m_{i+1} in order to get monotony without needing to change the spline on the previous intervals.

The demand that the spline is monotony or shape preserving does not fix the spline uniquely. The derivative parameters (a_i) and the breakpoint parameters (b_i) can still be chosen on certain parameter intervals. This freedom can be used to get the spline to fulfil (additional) requirements as well as possible. The parameters in our investigation will be fixed with an interactive iteration starting with values $a_i = 1$ and $b_i = 0.5$ for all values i. These values produce always a shape preserving spline.

Table 1. Distribution of sample trees by diameter and height classes.

d	Tree height, m																	Total
cm	1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	
	Number of trees																	
1	3	2																5
3		31	5															36
5		26	44	5														75
7			57	45	6													108
9			11	57	45	8	1											122
11			4	33	66	42	15											160
13			2	6	48	57	48	5										166
15				5	14	38	53	37	9									156
17					5	29	51	40	24	5								154
19				1	5	17	34	41	36	11	3							148
21						3	22	53	37	25	8							148
23							10	25	43	24	27	5						134
25							6	11	22	30	27	10						106
27							5	6	16	26	23	17	4					97
29								1	15	21	25	17	5	1				85
31							1	2	7	3	16	10	9	1				49
33									1	6	8	12	3	3	1			34
35							1		4	2	3	7	3	2				22
37											4	4	10	6	2			26
39											2	4	1	4	1			12
41											1		4	3				8
43												1	1	1	1			4
45														1	2	1		4
47												1			1			2
49																		
51																		
53												1			1			2
55																		
57																		
59																		
61																	1	1
Total	3	59	123	152	189	194	247	221	214	153	147	89	40	22	9	1	1	1 864

3. CONSTRUCTION OF THE TAPER CURVE

31. Background

The construction of a taper curve using spline functions is not a novel concept, it has been attempted e.g. by Sloboda (1976). Lahtinen and Laasasenaho (1979) thoroughly investigated the use of cubic splines for a taper curve. The result was that cubic splines gave a very good taper curve. Their most notable defect was a slight oscillation, the taper curve could be unmonotone even when the set of interpolating points was monotone. Consider for example the best taper curve with 15 interpolating points. In our sample tree material 24 % of monotone sets of interpolating points produced unmonotone parts to this taper curve (cf. the cubic spline CO15 in Figure 2). This oscillation also exaggerated the unmonotone parts of an unmonotone set of interpolating points (cf. the cubic spline CO15 in Figure 3).

A monotony preserving cubic spline which could give a taper curve with other good qualities has not yet been found. Therefore this study investigated the quality of taper curves constructed by monotony preserving quadratic splines. Another possibility to reduce the oscillation is to use non-polynomial splines (e.g. Späth 1983). The polynomial splines are however more convenient in calculations. Lahtinen and Laasasenaho (1979) showed that the usual interpolating quadratic spline was inferior to the interpolating cubic spline as a taper curve. The difference was not very great except at the butt where the quadratic spline could not cope with the rapid tapering of the stem. A monotony preserving quadratic spline is, however, much more flexible. Therefore it is possible that it can give a better taper curve than the interpolating cubic spline.

32. Sample tree material

We used the same sample tree material as Lahtinen and Laasasenaho (1979). All the material was collected for other purposes by the Finnish Forest Research Institute between 1968 and 1972. Only spruce stems were used in this study on the assumption that the resulting methods could be adapted for other species.

The localities (95) of this material were chosen from the survey tracts of the Finnish National Forest Inventory by random sampling. The material covers the whole Finland. The distribution of these sample trees into diameter- and length-classes is presented in Table 1.

Tree height was recorded to the nearest dm. Diameters were measured at fourteen different proportional heights from the ground, namely 1, 2.5 , 5, 7.5, 10, 15, 20, 30, 40, 50, 60, 70, 80 and 90 per cent. Moreover the diameter at 1.3 m was registered. The diameters were measured to the nearest mm by crosswise calipering. The height of the tree stump was also determined to the nearest cm. This was always at least 10 cm.

This sample tree material meets the demands of our investigation in relation to the measuring and selection very well. Its measurements give an adequate description of the actual taper curve and the material can be considered to be very representative. So it can be supposed that the monotony of this set of measurements gives a good approximation to the monotony of the tree in question. It can also be supposed that the convexity and concavity of the set of measurements give a reasonable approximation to the shape of the tree.

In addition of these 1864 spruce stems we also used a so called *normal tree*. Its height and diameters at proportional heights were defined as the mean values of the corresponding quantities in our sample tree material (see Figure 1).

Set-up 32.1
Normal tree: height 13.8 m, volume 173.2 dm^3

Rel. height:	1	2.5	5	7.5	10	15	20	30	40	50	60	70	80	90	100
Diameter:	25.1	21.3	19.2	18.3	17.8	17.2	16.6	15.3	13.9	12.2	10.4	8.3	6.0	3.4	0.4

33. Object of the investigation

We investigated the use of the monotony preserving quadratic spline as the taper curve. The investigation was divided into two parts.

The algorithm described in chapter 24 produces an interpolating quadratic spline which always preserves the monotony of the measurements. This spline is in this respect more suitable to a taper curve than the interpolating cubic spline which may produce oscillation. On the other hand the cubic spline is otherwise very suitable to a taper curve.

The primarily aim of the investigation was thus to find the extent to which the monotony preserving quadratic spline has the same good properties as the usual cubic spline as a taper curve. In this study the set of interpolating points consisted of all 14 measured diameters and the top diameter which was taken as 0.4 cm. The properties of the taper curve constructed by cubic splines are taken from Lahtinen and Laasasenaho (1979) which used the same sample tree material.

When the properties of the monotony preserving quadratic spline as a taper curve had been determined, we then turned to the second part of the investigation. This was to establish how much we could reduce the number of interpolating points in the monotony preserving spline without an apparent weakening of the properties of the taper curve. The quality of the taper curve depends on both the number, and placement of the interpolating points. There is not much choice in this study as far as placement is concerned, because we used certain measurements at fixed places. We did, however choose 8 different sets of interpolating points. The smallest sets contained only 4 points. The points in these 8 sets are presented in Table 2.

For all monotony preserving taper curves the construction was made by using the algorithm given in chapter 24 in the form of a FORTRAN-program.

34. Criteria of suitability

When examining the quality of a taper curve one has at first to decide which criteria are used. A normal demand is that the errors are small in all measurable quantities in some representative sample tree material. In addition to this one has to somehow estimate properties which are important but difficult to measure. One such property is how natural the shape of the taper curve is. All these inspections are made with regard to the sample tree material presented in chapter 32.

In the first part of the investigation taper curves were compared to the best taper curve in Lahtinen and Laasasenaho (1979) constructed by a cubic spline interpolating at 15 points. This taper curve will subsequently be called the *cubic taper spline* and denoted as CO15. It is often taken in the literature to be the right form of a tree (e.g. Lappi 1986).

The most important thing is that the taper curve quite accurately produces the volume of the stem or any part of it. Volumes are here always calculated making the assumption that the tree stem is a solid of revolution. We considered that a taper curve produced the right volume if it gave the same volume as the cubic taper spline CO15 for the whole stem, and for each of the seven parts into which we divided the stem. In the cases where CO15 was very unmonotonic some reservations had to be made. The division of stem and its effect on the normal tree are presented in Table 3.

From the sample tree material we evaluated the mean relative differences and their standard deviations by diameter and height classes. These differences were calculated for the total volume and for the afore-mentioned partial volumes.

The second criterion of suitability is the magnitude of diameter errors. Each taper curve was tested by evaluating the maximal diameter difference with regard of the cubic taper spline CO15 tree by tree. This difference was taken for the whole tree and for the subintervals used in volume estimation. The mean values were tabulated for each diameter class with the standard deviations. In addition trees with maximal diameter differences were tabulated.

Our third criterion of suitability is the form of the taper curve. This is an essential property needed e.g. in lumber assortment. Our quadratic splines preserve always the monotony of the measurements. Of course all monotony preserving curves do not give a natural form to a tree. It is, however difficult to measure deviation from the "right form". Therefore we chose representatives of the most typical tree forms from the sample trees. The graph of the taper curve was drawn for these trees. In addition the graphs of trees with interesting errors in diameters or volumes were drawn. These figures were a great help in detecting (hidden) weaknesses in taper curve models. Some typical graphs are presented in Figures 1–3.

The best taper curve constructed by a monotony preserving spline interpolating at 15 points will in the continuation be called the *monotony preserving taper spline* and denoted QO15.

In the second part of the investigation we examined the effect of the reduction of interpolating points to a monotony preserving taper curve. The criteria were the same as in the first part except that all comparisons were made with regard to the monotony preserving taper spline QO15. We also evaluated diameter errors and their standard deviations with regard to the measured diameters not used in interpolation. Some typical graphs used in the considerations of the forms of taper curves are presented in Figures 4–8.

FORTRAN-programs were created for all tree tests. We could make use of the programs of Lahtinen and Laasasenaho (1979) for some parts of the tests. The figures were drawn using the DISSPLA-program library.

Table 2. Percentage heights used in interpolation.

Point set	The percentage heights of interpolating points														
15	1	2.5	5	7.5	10	15	20	30	40	50	60	70	80	90	100
8A	1		5		10		20		40		60		80		100
8B		2.5		7.5		15		30		50		70		90	100
8C	1	2.5		7.5		15		30		50		70			100
5A	1			7.5			20				60				100
5B	1		5		10				40						100
5C		2.5		7.5			20				60				100
4A	1			7.5					40						100
4B		2.5			10					50					100

Table 3. The division of sample trees into parts for tests.

Number of part	Initial and terminal percentage heights of the part		The volume of the part in relation of the total volume (normal tree)
1	STUMP,	5	12 %
2	5,	10	11 %
3	10,	20	18 %
4	20,	40	29 %
5	40,	60	19 %
6	60,	80	9 %
7	80,	100	2 %

35. On the choice of parameters

The construction of a monotony preserving taper curve needs the following information:

Initial data:
- the number of interpolating points, n
- the relative heights $(x_i)_1^n$,
- the diameters at the relative heights, $(d_i)_1^n$,

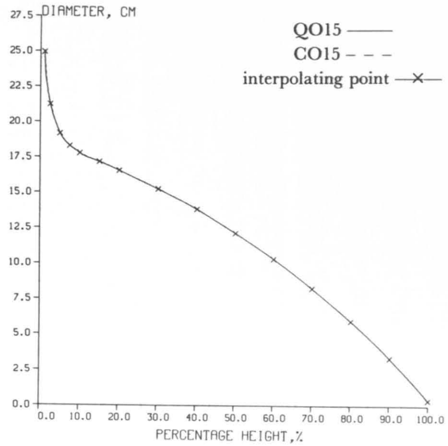

Figure 1. Taper curves QO15 and CO15 for the normal tree.

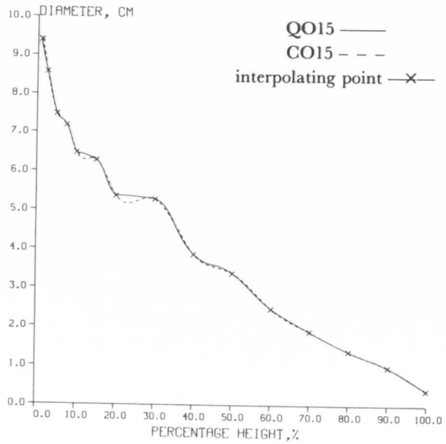

Figure 2. Taper curves QO15 and CO15 for a tree where measured diameters are monotone but CO15 is not.

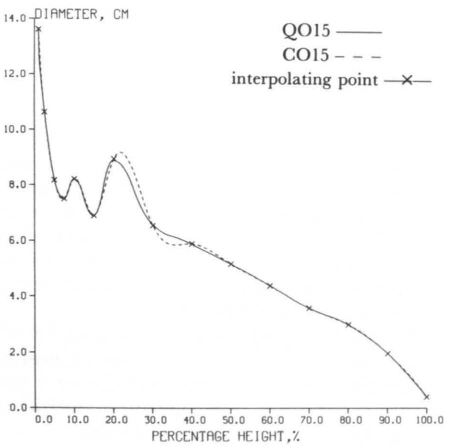

Figure 3. Taper curves QO15 and CO15 for an unmonotone tree.

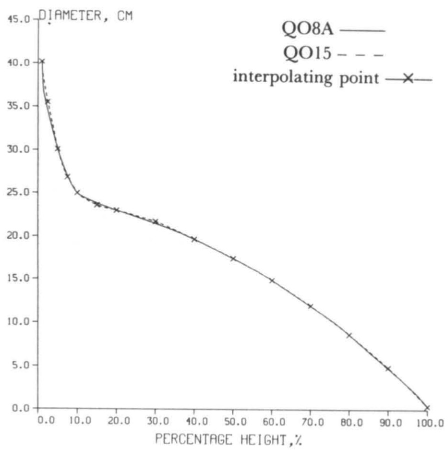

Figure 4. Taper curves QO8A and QO15 for a regular tree.

Parameters:
- the numbers $(a_i)_1^n$, determining derivatives of the spline at interpolating points
- the numbers (b_i), determining the places of additional breakpoints.

The initial data and the parameters have to be chosen so that the resulting taper curve fulfils our criteria as well as possible.

There is not much choice in the initial data because we use real measurements at fixed places. The number of interpolating points will be reduced in the second part of the investigation. De Boor (1978) has shown how much the placement of interpolating points may affect the accuracy of the approximation.

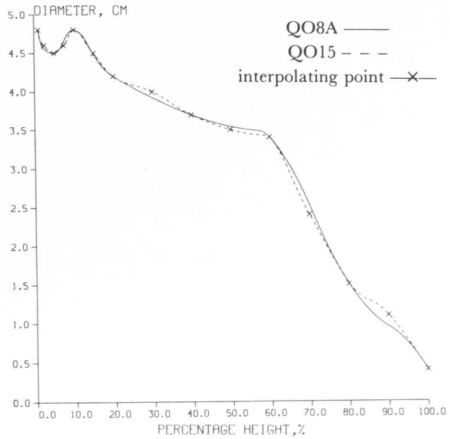

Figure 5. Taper curves QO8A and QO15 for an exceptional tree.

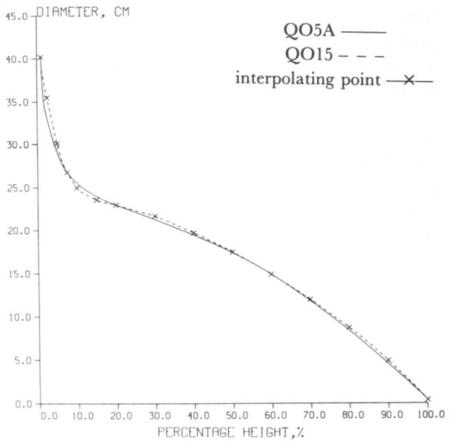

Figure 6. Taper curves QO5A and QO15 for a regular tree.

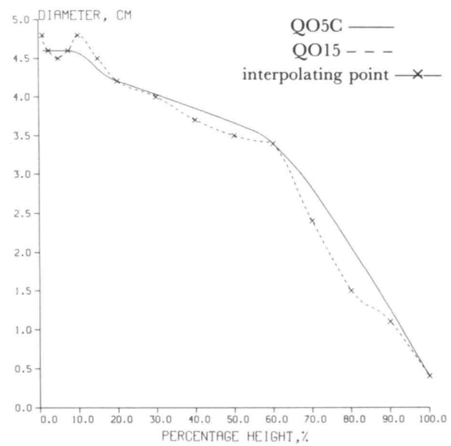

Figure 7. Taper curves QO5C and QO15 for an exceptional tree.

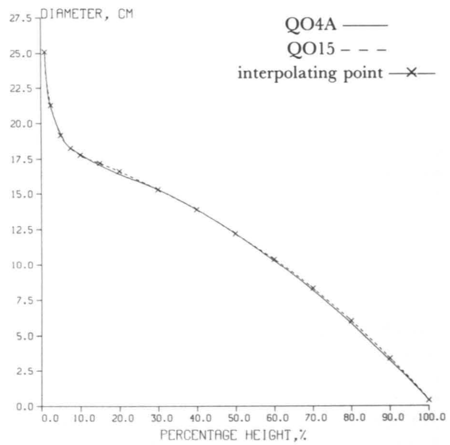

Figure 8. Taper curves QO4A and QO15 for the normal tree.

These results have little use in our case.

The parameters cannot be freely chosen. The algorithm of chapter 24 sets certain limitations. The numbers $(a_i)_1^n$ determine the derivatives of the spline at interpolating points and so the shape of the taper curve. In order to get a monotone curve we have to take $0 \leq a_i$ for each i. For the preservation of convexity and concavity we must set more strict limits (Lahtinen 1988). These limits depend on the interpolating points and measured diameters. Table 4 contains limits for the normal tree and the point set 8A (cf. Table 2).

In our algorithm a new breakpoint is added between two interpolating points whenever the derivatives at interpolating points do not fulfil the condition of alternative 31, $m_i + m_{i+1} = 2\delta_i$. Our choice of derivatives is such that nearly every interval gets a new breakpoint. For example when the point set 8A was used, then over 99 % of intervals got an additional breakpoint in our sample tree material.

The taper curve is monotony or shape preserving only if these additional breakpoints are taken in a certain subinterval, the so called ***breakpoint interval*** or *b-interval,* between interpolating points. The parameter b_i, $0 < b_i < 1$, determines the situation of the breakpoint on this subinterval. Table 5 gives information on the effect the placement of the breakpoint has on the volume of the stem. It appears that the effect is quite small. Thus by changing the place of breakpoints we can only obtain small corrections. Table 5 shows also that the more the taper curve changes the shorter the b-interval is.

In practice we firstly choose the number and places of interpolating points. Then the parameters (a_i) and (b_i) are determined for this point set with sufficient accuracy by using interactive iteration. As starting values we have $a_i = 1$ and $b_i = 0.5$ for each i. We first fix the parameters a_i and then make final adjustments with parameters (b_i). Iteration is easiest at the end points of the stem where a change affects only one interval. At all other points a change affects two intervals. Table 6 shows two examples of this kind of iteration in the minimization of the mean volume difference in all sample trees.

Parameters are always chosen so that the resulting taper curve is monotony preserving. If the preservation of the shape prevents a good volume estimate then we usually prefer the good volume estimate. Thus all taper curves are not necessarily shape preserving ones.

Table 4. The shape preserving limits of numbers (a_i) in the point set 8A for the normal tree.

i	Lower limit of a_i	Upper limit of a_i
1	0	∞
2	0.344	1.815
3	0.719	1.563
4	0	0.965
5	0.863	1.137
6	0.890	1.110
7	0.884	1.116
8	0	∞

Remark: With these limits the taper curve has exactly one point of inflection, namely the point x_4. In the interval $[x_1, x_4[$ the curve is convex and in the interval $]x_4, x_8]$ concave.

Table 5. The effect of the additional breakpoint to the mean volume.

I	length	ξ_{max}	ξ_{opt}	ξ_{min}	V_{min}	V_{max}	dV
STUMP−5	15	1.0	1.4	1.6	97.97	107.18	9.21
5−10	41	5.4	6.1	6.9	99.70	100.64	0.94
10−20	72	12.4	13.9	15.9	99.56	100.44	0.88
20−40	73	29.5	30.1	31.6	99.39	100.25	0.86
40−60	73	49.7	50.1	50.6	99.45	100.43	0.98
60−80	74	70.0	71.2	72.0	99.18	100.91	1.74
80−100	77	86.6	92.3	97.4	98.43	102.52	4.09

I — The interval of the stem
length — The percentage length of breakpoint interval in I
ξ_{max} — The place of additional breakpoint giving maximal mean volume in the interval
ξ_{opt} — The place of additional breakpoint giving the mean volume of CO15 in the interval
ξ_{min} — The place of additional breakpoint giving minimal mean volume in the interval
V_{min} — Minimal volume (volume of CO15 = 100) in the interval
V_{max} — Maximal volume (volume of CO15 = 100) in the interval.
dV — The difference of maximal and minimal volumes.

Remarks:
21.7 % of b-intervals were equal to the whole interval. $]x_i, x_{i+1}[$
48.2 % of b-intervals had the point x_i but not x_{i+1} as end point
31.3 % of b-intervals had the point x_{i+1} but not x_i as and point

Table 6. Iteration of some parameters in the minimization of the mean volume difference (compared with CO15).

Point set 5A, interval [stump, 5]

Step	a_1	a_2	b_1	b_2	dV	Standard deviation
1	1	1	0.5	0.5	3.53	5.39
2	3	1	0.5	0.5	2.63	5.10
3	3	0.9	0.5	0.5	1.81	4.96
4	3	0.9	0.6	0.5	1.09	4.81
5	3	0.9	0.7	0.5	0.36	4.67

dV = per cent mean volume difference on [stump, 5]

Point set 8A, interval [10,40]

Step	a_3	a_4	a_5	b_3	b_4	b_5	dV1	St.dev.	dV2	St. dev.
1	1	1	1	0.5	0.5	0.5	−0.07	2.11	−0.40	2.32
2	1	0.9	1	0.5	0.5	0.5	−0.02	2.10	−0.17	2.32
3	1	0.9	1	0.4	0.5	0.5	+0.02	2.07	−0.17	2.32
4	1	0.85	1	0.4	0.5	0.5	−0.04	2.09	−0.04	2.36

dV1 = per cent mean volume difference in [10,20]
dV2 = per cent mean volume difference in [20,40]

4. RESULTS

41. Monotony preserving quadratic spline as a taper curve

In the first part of the investigation we used all the measured diameters in the interpolation. In addition of these we used the fixed diameter 0.4 cm at the top. So we had 15 interpolating points at our disposal. Our aim was to examine if the taper curve constructed with a monotony preserving quadratic spline had the same good properties as the cubic taper spline CO15 (Lahtinen and Laasasenaho 1979).

We started with the volume of the stem. A monotony preserving quadratic spline was formed for each sample tree by using 15 interpolating points. The derivative parameters (a_i) and the breakpoint parameters (b_i) were determined so that the monotony preserving spline gave the same mean total volume for our sample trees as the cubic taper spline CO15. By interactive iteration we got the values of Set-up 41.1 for the parameters.

The taper curve obtained with these values is called the monotony preserving taper spline and denoted by QO15. It preserves the monotony for each sample tree and also the shape for most trees. The mean percentual volume differences of taper splines QO15 and CO15 in our sample tree material are to be found in Table 7. It shows that the monotony preserving taper spline QO15 really gives the same mean total volume as the cubic taper spline CO15. There is, however, a slight difference in partial volumes. This is at least partly due to the different monotony properties of taper splines (cf. Figures 1, 2 and 3). In our sample tree material the cubic taper spline CO15 was unmonotonic for 42.6 % of sample trees but monotony preserving taper spline QO15 was unmonotonic only for trees with an unmonotone set of measurements which consisted 24.7 % of sample trees (Lahtinen and Laasasenaho 1979). Also, as we said earlier, the monotony preserving spline and cubic spline have different behaviour in trees with an unmonotone set of measurements. The standard deviations of partial volume differences are, however, quite small.

There is in Table 7 also a taper spline QST15 which is formed by using the shape preserving quadratic spline with parameter values $a_i = 1$ and $b_i = 0.5$ for each i. A comparison of the taper splines QO15 and QST15 shows that our initial values gave quite good volume estimates except at the butt and top where there is a significant difference.

Table 8 shows the mean percentual total volume differences of the taper splines QO15 and CO15 in different diameter and height classes. The corresponding standard deviations are in Table 9. The differences are small and there is no apparent tendency so that the volume estimates of QO15 seem to be quite compatible with the ones of CO15.

Set-up 41.2 confirms that the monotony preserving taper spline QO15 gives practically the same total volumes as the cubic taper

Set-up 41.1

i	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
x_i	1	2.5	5	7.5	10	15	20	30	40	50	60	70	80	90	100
a_i	1.77	0.88	1	0.75	1.4	1	1	0.95	1	1	1	1	1	1.09	0.05
b_i	0.74	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	0.5	

Remark. Each parameter b_i determines a breakpoint on the breakpoint interval situated on the interval $]x_i, x_{i+1}[$.

Table 7. Mean total and partial volume differences and their standard deviations expressed as percentages.

Compared taper splines	Mean volume differences								Standard deviations							
	stump–100	stump–5	5–10	10–20	20–40	40–60	60–80	80–100	stump–100	stump–5	5–10	10–20	20–40	40–60	60–80	80–100
QO15–CO15	0.00	0.00	0.13	0.03	–0.01	–0.02	–0.03	–0.06	0.14	0.60	0.18	0.30	0.33	0.21	0.30	1.02
QST15–CO15	0.07	0.37	0.13	0.10	0.00	–0.02	–0.03	1.00	0.13	0.63	0.21	0.29	0.32	0.21	0.30	1.35
QO8A–QO15	–0.01	0.36	0.03	–0.06	0.03	0.04	0.08	0.24	1.17	4.67	2.07	2.06	2.31	2.59	4.01	11.43
QO8B–QO15	0.00	0.06	0.34	0.05	–0.03	–0.05	–0.20	–0.93	1.35	6.53	1.50	1.57	1.64	1.90	2.94	7.31
QO8C–QO15	–0.02	0.48	0.48	0.02	–0.01	0.01	0.00	0.90	1.21	2.11	1.54	1.57	1.64	1.89	3.07	23.93
QO5A–QO15	–0.02	1.69	0.55	0.02	0.01	–0.03	–0.19	0.24	2.14	7.03	1.66	2.74	3.81	4.45	7.16	28.44
QO5B–QO15	–1.00	0.07	–0.17	0.26	–1.80	–1.34	0.30	–0.44	3.33	4.63	2.07	4.08	6.45	5.00	7.27	28.47
QO5C–QO15	–0.21	0.00	0.17	–0.03	0.01	–0.03	–0.19	0.24	2.04	2.17	1.52	2.74	3.81	4.45	7.16	28.44
QO4A–QO15	–0.57	–0.23	0.21	–0.23	–0.40	–0.06	0.81	3.38	3.40	6.73	1.64	5.41	3.84	5.54	18.28	39.44
QO4B–QO15	–0.74	0.06	0.10	–0.51	0.34	–0.25	–4.31	5.18	3.20	4.98	3.26	3.80	5.58	2.30	15.78	42.32

spline CO15. The absolute values of per cent volume differences in the sample tree material were less than 0.2 % for 88.36 % of trees and the greatest volume difference was only 1.4 %. In fact the volume difference was over 1 % for only two trees.

Set-up 41.2
The distribution of absolute values of per cent total volume differences (QO15 -CO15).

dV (%)	0.2	0.4	0.6	0.8	1.0	1.2	1.4
trees (%)	88.36	98.55	99.62	99.84	99.89	99.95	100

trees (%) = per cent amount of sample trees having absolute value of per cent total volume difference less than dV.

In addition to volume differences diameter differences were also calculated. Table 10 shows mean maximal diameter differences in centimeters between the monotony preserving taper spline QO15 and the cubic taper spline CO15. The differences are tabulated by diameter classes for the whole tree and for the partial intervals.

The differences are in general small and at least partly due to the different monotony properties of the curves (see Figures 1, 2 and 3). The oscillation of the cubic taper spline takes place in most cases on the interval [5, 40]. On the stable part of the stem which is from 40 % to 80 % there is very little unmonotony. The mean maximum diameter difference in this part is therefore only 0.4 mm (Table 10).

At the butt one source of differences is the height of the tree stump. Especially for tall trees the height of the stump is less than the lowest interpolating point which is 1 % of the total height. The taper curve is used in the interval from the stump to the top. This means that the taper curve is evaluated under the 1 % height by extrapolation. The derivative at the height of 1 % is usually very steep and this may produce quite large diameter values in extrapolation. In fact, in the interval [stump, 1] the greatest diameter difference is 28.4 cm but in the interval [1, 2.5] only 2.6 cm (see Table 13).

Figures 1, 2 and 3 give an impression of how naturally the monotony preserving taper spline QO15 behaves. Figure 1 shows the normal tree as calculated by the monotony preserving taper spline and by the cubic taper spline CO15. There is no visible difference between these two curves. Figure 2 shows a situation where the interpolating point set is monotone, but cubic taper spline is not. The monotony preserving taper spline behaves naturally here as is to be expected. Figure 3 is an example of how different monotony preserving taper spline QO15 and cubic taper spline CO15 may be for an unmonotone tree.

Table 8. Mean per cent total volume differences of QO15 and CO15 by diameter and height classes.

diameter (cm)	height (m) 1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	mean value
1	0.07	–0.18																–0.03
3		0.12	0.03															0.11
5		0.11	0.07	0.07														0.08
7			0.04	0.02	0.01													0.03
9			0.06	0.06	0.00	–0.06	0.10											0.03
11			–0.05	0.00	0.00	–0.03	–0.01											–0.01
13			0.19	0.00	–0.01	0.00	–0.01	–0.07										–0.01
15				0.00	0.00	–0.01	0.00	–0.01	–0.01									–0.01
17					0.02	0.05	–0.01	0.00	0.00	–0.02								0.00
19				–0.11	–0.04	0.01	–0.03	0.00	–0.02	–0.05	0.28							–0.01
21						0.16	–0.05	0.01	–0.01	–0.04	–0.04							–0.01
23							–0.03	0.01	–0.02	–0.02	–0.02	–0.02						–0.02
25							–0.01	0.01	–0.03	–0.03	–0.05	–0.05						–0.03
27							0.02	0.02	–0.01	0.02	–0.02	–0.02	–0.03					0.00
29								–0.05	–0.04	–0.02	–0.01	–0.10	–0.01	0.53				–0.03
31							–0.01	0.01	0.06	0.05	–0.04	–0.04	–0.04	0.04				–0.02
33									0.00	0.07	–0.07	0.02	–0.09	–0.01	–0.13			–0.01
35							–0.04		0.00	–0.04	0.02	0.02	–0.09	–0.07				–0.02
37											0.16	0.00	–0.07	–0.07	0.29			0.00
39											–0.08	0.02	0.29	–0.06	–0.05			0.00
41											0.05		–0.11	0.02				–0.04
43												0.14	–0.14	0.13	0.16			0.08
45														–0.06	0.00	–0.02		–0.02
47												–0.12			–0.20			–0.16
49																		
51																		
53												0.00			–0.27			–0.13
55																		
57																		
59																		
61																	0.01	0.01
Mean value	0.07	0.11	0.05	0.03	0.00	0.00	–0.02	0.00	–0.02	–0.02	–0.02	–0.03	–0.05	0.00	0.01	–0.02	0.01	0.00

The monotony preserving spline is clearly more stable in the unmonotone parts than the cubic spline, which concentrates on the minimization of curvature.

Theoretically a drawback in the monotony preserving quadratic spline is that it may behave too "angularly" at the butt. One reason is that the quadratic spline has only one continuous derivative. Another fact is that the parabolas of a quadratic spline can only for a short interval follow the fast tapering on the butt which is logarithmic in nature. The cubic spline is better in this context. In practice this difference is not to be seen in the case of 15 interpolating points.

The taper spline QO15 preserves always the monotony of the measurements and in most cases also the shape. The preference of the volume estimation induces sometimes such values to the derivative parameters that the resulting spline does not preserve the shape for every tree.

Table 9. Standard deviations of per cent total volume differences (QO15–CO15) by diameter and height classes.

diameter (cm)	height (m) 1	3	5	7	9	11	13	15	17	19	21	23	25	27	29	31	33	mean value
1	0.15	0.04																0.18
3		0.27	0.26															0.27
5		0.21	0.20	0.19														0.20
7			0.15	0.26	0.12													0.20
9			0.16	0.14	0.11	0.12	0.00											0.13
11			0.21	0.16	0.14	0.10	0.12											0.14
13			0.30	0.16	0.12	0.15	0.10	0.20										0.13
15				0.30	0.13	0.15	0.11	0.10	0.07									0.13
17					0.16	0.17	0.12	0.09	0.10	0.09								0.12
19				0.00	0.13	0.17	0.13	0.10	0.10	0.09	0.61							0.14
21						0.28	0.18	0.11	0.11	0.11	0.05							0.13
23							0.16	0.12	0.12	0.09	0.11	0.08						0.11
25							0.12	0.16	0.12	0.07	0.10	0.05						0.10
27							0.11	0.05	0.09	0.09	0.14	0.11	0.15					0.11
29								0.00	0.08	0.09	0.11	0.07	0.09	0.00				0.11
31							0.00	0.04	0.27	0.15	0.10	0.08	0.11	0.00				0.13
33									0.00	0.13	0.09	0.11	0.13	0.03	0.00			0.11
35							0.00		0.19	0.01	0.11	0.15	0.15	0.01				0.13
37											0.12	0.16	0.08	0.07	0.34			0.16
39											0.00	0.13	0.00	0.09	0.00			0.13
41											0.00		0.09	0.12				0.11
43												0.00	0.00	0.00	0.00			0.14
45														0.00	0.10	0.00		0.06
47												0.00			0.00			0.05
49																		
51																		
53												0.00			0.00			0.19
55																		
57																		
59																		
61																	0.00	0.00
Mean value	0.15	0.25	0.18	0.19	0.13	0.15	0.12	0.11	0.11	0.10	0.14	0.11	0.11	0.14	0.23	0.00	0.00	0.14

The treatment of the top forms an exception. There the derivative values producing the best volume estimate often produces an extra point of inflection to the interval [90, 100]. As this part contains only 2% of the total volume we prefer to have a good shape rather than an optimal volume estimate.

In this connection we may mention that our initial choice of derivatives was so natural that the algorithm had to change the derivative (alternative 32, case 3223) only in 1.75 % of all cases (Table 11).

In conclusion we can say that the monotony preserving quadratic spline gives a very good taper curve. It preserves the monotony of the measurements and behaves very naturally. It gives reliable volume and diameter estimates. The monotony preserving taper spline QO15 is quite compatible with the cubic taper spline CO15 and surpasses it in aspects concerning the monotony properties.

Table 10. Mean maximal diameter differences (cm) of QO15 and CO15 by diameter classes in different parts of the stem.

diameter (cm)	Intervals of the stem stump−100	stump−5	5−10	10−20	20−40	40−60	60−80	80−100
1	0.06	0.02	0.03	0.05	0.03	0.02	0.01	0.02
3	0.08	0.04	0.02	0.04	0.04	0.03	0.02	0.02
5	0.09	0.06	0.03	0.05	0.05	0.02	0.02	0.03
7	0.13	0.11	0.03	0.06	0.05	0.02	0.02	0.03
9	0.19	0.18	0.04	0.05	0.05	0.02	0.02	0.03
11	0.24	0.24	0.04	0.06	0.05	0.03	0.03	0.03
13	0.32	0.31	0.05	0.08	0.07	0.03	0.03	0.04
15	0.40	0.39	0.05	0.07	0.07	0.03	0.03	0.05
17	0.48	0.47	0.06	0.09	0.07	0.04	0.04	0.05
19	0.81	0.81	0.06	0.09	0.08	0.04	0.04	0.05
21	0.54	0.53	0.07	0.10	0.09	0.05	0.05	0.06
23	0.80	0.79	0.06	0.09	0.09	0.04	0.05	0.05
25	0.58	0.58	0.08	0.10	0.09	0.05	0.05	0.07
27	0.86	0.86	0.09	0.11	0.09	0.06	0.06	0.09
29	0.88	0.87	0.09	0.12	0.11	0.06	0.05	0.06
31	0.76	0.75	0.11	0.13	0.11	0.06	0.08	0.08
33	0.83	0.81	0.10	0.15	0.14	0.06	0.07	0.09
35	1.08	1.07	0.11	0.16	0.15	0.07	0.05	0.08
37	1.96	1.95	0.13	0.15	0.16	0.07	0.06	0.08
39	1.79	1.79	0.11	0.19	0.14	0.07	0.07	0.07
41	0.85	0.85	0.17	0.18	0.15	0.08	0.12	0.12
43	1.03	1.03	0.21	0.18	0.26	0.07	0.07	0.10
45	1.96	1.93	0.18	0.15	0.08	0.08	0.09	0.08
47	1.04	1.04	0.15	0.19	0.14	0.07	0.04	0.15
49								
51								
53	1.61	1.61	0.23	0.17	0.14	0.05	0.01	0.12
55								
57								
59								
61	0.78	0.78	0.31	0.07	0.10	0.11	0.18	0.20
Mean	0.53	0.52	0.06	0.09	0.08	0.04	0.04	0.05

42. Taper curve through seven measured diameters

We have shown that when 14 measured diameters are used then the monotony preserving quadratic spline gives a taper curve with all the necessary qualities. After this it is natural to ask can we construct essentially as good a monotony preserving taper curve using fewer measured diameters. The study of this question forms the second part of the investigation. Our method is to form monotony preserving quadratic splines interpolating in a set containing less than 15 points and to examine how well these splines can approximate the monotony preserving taper spline QO15.

We started the investigation with eight interpolating points. Three different combinations, called 8A, 8B and 8C were chosen as Table 2 shows. Each of them contained 7 measured diameters and the fixed top diameter. Monotony preserving quadratic splines were construted by using diameters at the chosen points. The derivative parameters (a_i) and breakpoint parameters (b_i) were determined by interactive iteration so that the resulting taper curve was as good an approximation to QO15 as possible. The iteration was interrupted when the parameter changes improved the taper curve only marginally. The resulting taper curves were called again *monotony preserving taper splines* and denoted by QO8A, QO8B and QO8C. Set-up 42.1 shows the values of their parameters. The derivatives were again so natural that the algorithm had to change them very seldom in order to guarantee the monotony (Table 11).

Table 11. The derivatives changed by the algorithm (case 3223) in order to preserve monotony.

Interpolating point x_i:	1	2.5	5	7.5	10	15	20	30	40	50	60	70	80	90	100	①	②	③
Number of changed derivatives at x_i for point set 15	0	7	23	64	99	103	119	30	16	7	2	2	4	13	0	489	450	1.75
Number of changed derivatives at x_i for point set 8A	0		1		40		68		0		0		0		0	109	109	0.73
Number of changed derivatives at x_i for point set 5C		0		5			41				0				0	46	46	0.49

① Total number of changed derivatives
② Number of trees with changed derivatives
③ Per cent number of derivative changes

Set-up 42.1
Taper spline QO8A

index i	1	2	3	4	5	6	7	8
rel.height x_i	1	5	10	20	40	60	80	100
der.param. a_i	3.0	0.9	1	0.83	1.01	1.03	1.05	0.19
b-point pm. b_i	0.7	0.5	0.35	0.5	0.5	0.35	0.5	

Taper spline QO8B

index i	1	2	3	4	5	6	7	8
rel.height x_i	2.5	7.5	15	30	50	70	90	100
der.param. a_i	1	0.9	1	1	1.2	1.3	1	0.8
b-point pm. b_i	0.53	0.5	0.5	0.5	0.5	0.5	0.5	

Taper spline QO8C

index i	1	2	3	4	5	6	7	8
rel.height x_i	1	2.5	7.5	15	30	50	70	100
der.param. a_i	2.2	1.8	1	1	0.95	1	1.2	0.09
b-point pm. b_i	0.95	0.5	0.5	0.5	0.5	0.5	0.2	

We first considered the volume errors of these three taper splines. The mean total volume in our sample tree material can be adjusted to be practically the same as the one of QO15 by choosing derivative parameters (a_i) accordingly. However, the mean partial volume errors remain away from zero as Table 7 shows.

The monotony preserving taper spline QO8A does not express the butt (from stump to 5 % height) quite naturally but otherwise volume errors are small. In the point set 8B the lowest height is 2.5 %. This effects that the corresponding taper spline QO8B has a very small mean volume error on the butt. Unfortunately the mean volume error on the next interval, [5, 10], is respectively higher. Point set 8C attempts to eliminate problem with the butt by having an additional interpolating point there. This results in clear failure. The corresponding taper spline QO8C has greater volume errors on the butt than the other two alternatives.

Figure 9 shows the distribution of the absolute values of per cent total volume errors of

Table 12. Mean maximum diameter differences (cm).

Taper splines	Stump −100	Stump −5	5−10	Interval 10−20	20−40	40−60	60−80	80−100
QO15−CO15	0.53	0.52	0.06	0.09	0.08	0.04	0.04	0.05
QST15−CO15	0.46	0.39	0.07	0.08	0.08	0.04	0.04	0.24
QO8A−QO15	0.88	0.79	0.25	0.22	0.23	0.21	0.23	0.32
QO8B−QO15	1.34	1.28	0.40	0.33	0.32	0.32	0.39	0.35
QO8C−QO15	0.90	0.70	0.42	0.33	0.32	0.30	0.40	0.49
QO5A−QO15	1.14	1.00	0.51	0.35	0.38	0.37	0.47	0.55
QO5B−QO15	1.06	0.81	0.25	0.44	0.59	0.45	0.48	0.57
QO5C−QO15	1.37	1.23	0.40	0.35	0.38	0.37	0.47	0.55
QO4A−QO15	1.26	1.01	0.47	0.54	0.47	0.50	0.74	0.70
QO4B−QO15	1.69	1.52	0.46	0.44	0.57	0.54	0.83	0.77

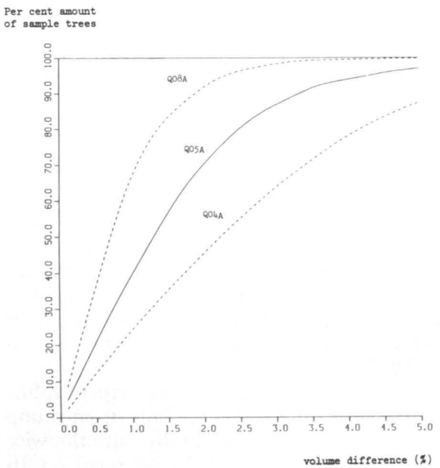

Figure 9. Per cent amount of sample trees with absolute values of per cent total volume differences less than a given number. (Taper curves are compared with QO15.)

the taper spline QO8A into magnitude classes. Although the mean total volume error is zero, only 16 % of sample trees have the absolute per cent total volume error less than 0.2 %. However, two thirds of sample trees had total volume errors less than 1 % when the taper spline QO8A was used and only four trees had total volume error exceeding 5 %. The other two 8-point monotony preserving taper splines have essentially the same distributions as QO8A.

The mean maximal diameter differences between 8-point taper splines and QO15 for the whole sample tree material are shown in Table 12, the extreme cases are in Table 13. The greatest differences occur again on the butt. From the height 5 % upwards the mean maximal diameter differences are small: 0.3 cm for QO8A, 0.4 cm for QO8B and 0.5 cm for QO8C.

On the whole we can say that the diameter differences between 8-point taper splines and taper spline QO15 are relatively small. This is also true for comparisions with measured diameters (see Tables 14 and 15). On the butt the taper spline QO8B behaves worse than the other two alternatives. This is again due to the fact that QO8B extrapolates from height 2.5 % downwards. The taper spline QO8A has the smallest diameter differences.

These 8-point taper splines preserve always the monotony of measurements and mostly also the shape. Among these three taper splines the alternative QO8A seems to have the most natural shape although the other two possibilities are also quite acceptable. In fact, for the normal tree there is practically no visible difference between QO8A and QO15. For trees with a regular shape the situation is almost the same (Figure

Table 13. Maximal diameter differences in all the sample tree material (cm).

Taper spline	Stump−5	5−10	10−20	Interval of the stem 20−40	40−60	60−80	80−100
QO15	28.84 (2.64)	0.51	0.71	0.72	0.28	1.02	0.74
QO8A	26.15 (6.54)	2.61	2.59	2.65	1.70	1.99	4.93
QO8B	84.59 (2.27)	2.34	2.63	2.64	1.74	4.85	5.00
QO8C	59.37 (2.54)	2.57	2.62	2.63	1.77	3.68	4.44
QO5A	26.95 (8.80)	3.67	2.77	2.75	2.81	4.09	4.48
QO5B	26.17 (6.27)	2.80	4.38	3.98	3.26	4.17	4.54
QO5C	24.94 (2.24)	2.34	2.68	2.75	2.81	4.09	4.48
QO4A	26.98 (7.62)	2.98	4.61	3.77	3.07	5.83	5.86
QO4B	34.80 (4.21)	4.42	3.82	3.81	2.50	6.08	6.11

Remark:
The quantities in brackets show the maximum diameter difference from the lowest interpolating point to the 5 % of total height.

Table 14. The mean diameter error with regard to the measured diameters (cm).

Taper spline	Relative height (%) 1	2½	5	7½	10	15	20	30	40	50	60	70	80	90
QO8A		0.09		0.04		0.00		−0.01		0.00		0.00		−0.04
QO8B	−0.61		0.11		0.02		−0.02		0.01		0.02		−0.18	
QO8C			0.17		0.01		−0.03		−0.02		0.05		−0.13	−0.11
QO5A		0.12	0.26		0.00	0.00		0.00	−0.02	0.01		−0.04	−0.14	−0.18
QO5B		0.04		0.00		0.03	−0.05	−0.19	−0.18	−0.07		−0.01	−0.14	−0.20
QO5C	−0.35		0.07		−0.01	0.00		0.00	−0.02	0.01		−0.04	−0.14	−0.18
QO4A		−0.12	0.07		0.05	−0.06	−0.12	−0.03		−0.03	−0.08	−0.12	−0.17	−0.19
QO4B	0.68		−0.05	0.03		−0.05	−0.12	0.01	0.19		−0.36	−0.32	−0.24	−0.12

Table 15. Standard deviations of diameter errors with regard to the measured diameters.

Taper spline	Relative height (%) 2.5	5	7.5	10	15	20	30	40	50	60	70	80	90
QO8A	0.86		0.33		0.31		0.31		0.28		0.30		0.42
QO5A	1.03	0.56		0.35	0.33		0.38	0.40	0.34		0.30	0.53	0.57
QO4A	1.02	0.53		0.40	0.53	0.53	0.53		0.41	0.58	0.72	0.77	0.66

4). Figure 5 shows how well the monotony preserving taper spline QO8A can reproduce a quite unmonotone tree.

In conclusion we may say that all these three monotony preserving taper splines approximate QO15 quite well and are thus acceptable taper curves. Alternative QO8A is the best of these three taper splines.

This study of 8-point taper splines provided evidence of the fact that the monotony preserving quadratic spline can produce a better taper curve than the usual cubic spline. Table 16 measures with mean volume differences how well an 8-point taper spline can reproduce a 15-point taper spline. It appears that the monotony preserving taper spline

QO8A approximates the monotony preserving taper spline QO15 better than the corresponding 8-point cubic taper spline CO8A approximates the cubic taper spline C015. In the former case the mean total volume difference is -0.01 % and in the latter 0.15 %.

43. Taper curve through four measured diameters

In the previous section we saw that the monotony preserving taper spline with seven measured diameters had essentially the same qualities as that with 14 measured diameters. This encouraged us to reduce the number of measured diameters further. The taper splines were again constructed to approximate the monotony preserving taper spline QO15.

We chose three different five point combinations, namely the point sets 5A, 5B and 5C defined in the Table 2. Each of them contained 4 measured diameters and the fixed top diameter. Monotony preserving quadratic splines were constructed by using diameters at the chosen points. Values for parameters (a_i) and (b_i) were determined by interactive iteration which was continued as long as there was a clear improvement in the taper spline. The final parameter values determined monotony preserving taper splines QO5A, QO5B and QO5C and are presented in Set-up 43.1 (see also Table 11 about the naturality of derivatives).

Set-up 43.1

Taper spline QO5A

index i	1	2	3	4	5
rel. height x_i	1	7.5	20	60	100
der. param. a_i	3.5	0.65	0.72	1	0.105
b-point pm. b_i	0.5	0.5	0.5	0.5	

Taper spline QO5B

index i	1	2	3	4	5
rel. height x_i	1	5	10	40	100
der. param. a_i	3	0.9	0.8	1	0.095
b-point pm. b_i	0.74	0.3	0.2	0.5	

Taper spline QO5C

index i	1	2	3	4	5
rel. height x_i	2.5	7.5	20	60	100
der. param. a_i	1.15	1	0.72	1	0.105
b-point pm. b_i	0.5	0.5	0.5	0.5	

We again start with volume errors which are presented in Table 7. For the monotony preserving taper spline QO5A the mean percentual total volume is practically the same as for the taper spline QO15. The mean percentual partial volume errors remained, however, quite large at the butt. The small number of interpolating points apparently prevents better results. The taper spline QO5A gives quite good volume estimates on the interval from 20 % height to 60 % height.

The point set 5B is more concentrated to the butt than the previous set. The mean volume error of the monotony preserving taper spline QO5B is also much smaller at the butt than the one of the taper spline QO5A. On the other hand the mean volume error in the interval [20,40] is much bigger than before. On the whole the taper spline QO5B is weaker in volume predictions than the taper spline QO5A. This can also be seen in standard deviations. Thus we cannot recommend the use of the alternative QO5B.

Lifting the lowest interpolating point in point set 5A to the height of 2.5 % produces the monotony preserving taper spline QO5C. It gives clearly better volume estimates for the butt than the taper spline QO5A without any evident negative effect to other parts. The mean total volume error is, however, slightly greater for QO5C than QO5A even if the standard deviations are smaller.

Figure 9 shows that the absolute values of relative total volume errors of the taper spline QO5A do not have as good a distribution in the magnitude classes as the taper spline QO8A. Now 40% of sample trees had absolute total volume error less than 1%. The taper splines QO5B and QO5C have approximately similar distributions as QO5A.

The mean maximal diameter differences for all the sample tree material are shown in Table 12 and the extreme cases in Table 13. At first sight it appears that the mean maximal diameter differences are only little higher than for 8-point splines. In practice this tells us only the situation at the butt because the maximal diameter difference nearly always occurs at the butt. On examining the mean maximal diameter differences at other parts of the stem it is obvious that they are larger than in the case of 8 points. However, from the height 5 % upwards the mean maximal diameter differences are for all three taper splines less than 0.6 cm. The mean errors with regard to measured diameters are still small (mostly under 0.1 cm) for all 5-point taper splines i.e. QO5A, QO5B and QO5C (Tables 14 and 15).

These 5-point taper splines preserve always the monotony of the measurements and mostly also the shape. On examining the shape of the graphs of these taper splines it is obvious that alternative QO5B is the worst one even on this respect. Taper splines QO5A and QO5C behave in much the same manner. This is not surprising as they have four common interpolating points. For the normal tree the taper curves QO5A and QO15 are almost identical and the differences are still small for trees with a regular shape (Figure 6). If the stem is somehow exceptional then the 5-point spline may differ quite a lot from the taper spline QO15 even though it may have a nice shape (Figure 7).

In conclusion we can say that in most cases the monotony preserving taper splines QO5A and QO5C generate satisfactory approximations to the taper spline QO15 and are thus satisfactory taper curves.

Table 16 provides more evidence of the fact that the monotony preserving taper spline suffers less than the cubic spline of the reduction of the number of interpolating points. It measures volume differences and shows that the five point monotony preserving taper spline QO5A gives a better approximation to the monotony preserving taper spline QO15 (mean total volume error -0.02 %) than the 8-point cubic taper spline CO8A gives to the cubic taper spline CO15 (mean total volume error 0.15 %). The standard deviations are greater in the 5-point case, however.

Table 16. Mean per cent volume differences due to the reduction of interpolating points for cubic and quadratic taper splines.

Interval	Mean per cent volume difference			Standard deviation		
	CO8A–CO15	QO8A–QO15	QO5A–QO15	CO8A–CO15	QO8A–QO15	QO5A–QO15
STUMP, 100	0.15	–0.01	–0.02	1.10	1.17	2.14
STUMP, 5	–1.9	0.4	1.7	2.0	4.7	7.0
5, 10	–0.4	0.0	0.5	2.3	2.1	1.7
10, 20	0.3	–0.1	0.0	2.8	2.1	2.7
20, 40	–0.3	0.0	0.0	3.1	2.3	3.8
40, 60	0.2	0.0	0.0	3.0	2.6	4.4
60, 80	0.1	0.1	–0.2	4.5	4.0	7.2
80, 100	–0.3	0.2	0.2	2.3	11.4	28.4

44. Taper curve through three measured diameters

Finally we made some experiments with four interpolating points. These were the sets 4A and 4B in Table 2. Both contained three measured diameters and the fixed top diameter. The corresponding monotony preserving taper splines QO4A and QO4B were determined as approximations to the taper spline QO15 using the same principles as those used with previous approximations. Interactive iterations were carried on, however, only some steps. The resulting taper splines are therefore not necessarily the best approximations of QO15. They were none the less able to expose the general quality of taper splines with four interpolating points. The parameters of taper splines QO4A and QO4B are in Set-up 44.1.

Set-up 44.1
Taper spline QO4A

index i	1	2	3	4
rel. height x_i	1	7,5	40	100
der.param. a_i	3.7	0.65	0.82	0.18
b-point pm. b_i	0.5	0.5	0.5	

Taper spline QO4B

index i	1	2	3	4
rel.height x_i	2.5	10	50	100
der.param. a_i	2.5	0.8	1.2	0.2
b-point pm. b_i	0.57	0.47	0.3	

Table 7 shows that the mean total volume error is for both four point taper splines clearly larger than the errors for the five point taper splines QO5A and QO5C. The error is not very large at the butt, especially in the case of taper spline QO4B. In the middle parts of the stem the results are undoubtly inferior to the monotony preserving taper splines with more interpolating points. The standard deviations are also slightly larger.

Figure 9 shows the same tendency in the distribution of the absolute total volume errors into magnitude classes. The four point taper spline QO4A is weaker than others. Only 25 % of sample trees have absolute total volume error less than 1%. The taper spline QO4B behaves essentially in the same way as QO4A.

Table 12 shows the mean maximal diameter differences in sample trees. They are larger for four point taper splines than for five point taper curves but are from the height 5 % upwards less than 0.9 cm. The maximal diameter differences are a little higher than for five point taper splines (Table 13). The differences to the measured diameters are also greater than the ones of five point taper splines, but are mostly less than 0.2 cm (Tables 14 and 15).

Also the 4-point taper splines preserve always the monotony of the measurements. Figure 8 shows that although taper spline QO4A gives a satisfactory approximation to the taper spline QO15 for normal tree, there is already systematic error in the graph. If the tree is irregular then the four point taper spline does not have much chance of modelling it.

In conclusion we may say that the monotony preserving quadratic spline produces an agreable shape to the four point taper spline, but the measurable qualities are clearly weaker than in the case of five points.

5. CONCLUSIONS

51. Monotony preserving quadratic spline as a taper curve

When there are several measured diameters (and the height) available for a tree, the perhaps best taper curve has been constructed to date using a cubic spline. The cubic spline has, however, a weakness, this being a slight oscillation which may result in an unmonotone taper curve for a monotone tree. This oscillation is due to intrisic properties of the cubic spline and cannot therefore be totally removed.

A quadratic spline can be constructed so that it preserves the monotony of a given set of interpolating points. The algorithm for the construction of such a spline contains several parameters which have to be chosen within certain limits. The freedom in the choice of parameters can be used to get the spline to fulfil additional restrictions. The algorithm needs no matrix inversions. It has essentially the same degree of complexity and requires a similar number of calculations as the algorithms used in the construction of a cubic spline. The algorithm can be run in a computer of PC-type.

In the first part of our investigation we examined how suitable the monotony preserving quadratic spline is for a taper curve when several measured diameters and the height are available. For this purpose we used comparisions with the cubic taper spline CO15 which was the best taper curve in Lahtinen and Laasasenaho (1979).

The preservation of monotony is an advantage of the quadratic spline over the cubic spline. The results of chapter 41 show that in other respects the monotony preserving quadratic spline has as a taper curve similar good qualities as the cubic spline. The taper spline QO15 gives as reliable volume estimates as the best cubic taper spline CO15. The diameter estimates of the taper spline QO15 are even more reliable than the ones of CO15 because of the monotony.

At the butt the monotony preserving quadratic spline is in theory a little stiffer than the cubic spline. This arises from the smaller degree of the polynomial pieces. The greater number of breakpoints of the monotony preserving spline tends to compensate this stiffness. In case of 15 interpolating points this compensation is sufficient, but with a distinctly lesser amount of interpolating points there may be differences.

Another theoretical difference is that the monotony preserving quadratic spline may sometimes have a more angular shape than the usual cubic spline. The reason for this is that the quadratic spline has only one continuous derivative while the cubic spline has two. On the other hand the oscillation of the cubic spline is very much due to the continuity of the second derivative. In practice this angularity is not to be seen in the case of 14 measured diameters. With less than 8 measured diameters it may exist but only slightly.

The greatest difference in favour of the taper spline QO15 is that it preserves always the monotony of the measurements and mostly also the shape. This makes the monotony preserving taper spline more reliable e.g. in lumber assortment and in growth studies.

If a tree has a regular shape then the monotony preserving taper spline QO15 and the cubic taper spline CO15 give similar results and are in this sense equal. For other trees there are differences in favour of the monotony preserving taper spline. Especially it is to be noticed that it is difficult to know beforehand when the cubic taper spline will oscillate whereas the monotony properties of the quadratic taper spline are always known.

The monotony preserving taper spline is a local spline. This means that it has many parameters each of which affects the spline only locally. Here lies the strength of this spline and also its weakness. Many local parameters mean that we can take much information into account in the construction of the taper spline. For instance if it is known that the tree is unmonotone at a certain height, then the taper curve can be constructed to be similarly unmonotone even if this unmonotony is not to be seen in the measurements.

The weakness of the local spline is that the many parameters has to be determined even if there is very little information available. This means some kind of estimation of parameters. Our algorithm gives limits for these parameters as well as certain initial values. These initial values are quite good estimates except on the butt and top.

The choice of parameters of the monotony preserving taper spline QO15 was made by using a very representative sample tree material. This means that QO15 is reliable for the treatment of any tree with diameter measurements on the appropriate heights.

In conclusion we can say that with a monotony preserving quadratic spline it is possible to get a better taper curve than with a cubic spline.

52. The number of measured diameters in the taper spline

The first part of investigation showed that the monotony preserving quadratic spline gives an accurate taper curve when 14 measured diameters are used. After this it was natural to examine whether we could construct an essentially as good taper curve with a monotony preserving quadratic spline with fewer measured diameters. For this purpose we investigated how well we could approximate the monotony preserving taper spline QO15 with a monotony preserving quadratic spline interpolating in a smaller set.

When seven measured diameters were used then the monotony preserving taper spline QO8A was quite good approximation to the taper spline QO15. They have in our sample tree material the same mean total volumes and there are in partial volumes differences only at the butt. Also the differences in diameter and shape are small. The differences on the butt are mostly due to the stiffness and angularity mentioned in the chapter 51. At other parts of the taper curve these phenomena are not to be seen (cf. Figure 4). On the whole the eight point monotony preserving taper spline is still a good taper curve.

The monotony preserving taper splines QO5A and QO5C with four measured diameters still offer satisfactory approximations to the taper spline QO15. The mean total volume is in our sample tree material practically the same as for the taper spline QO15 but there are differences in the mean partial volumes. The diameter differences were greatest on the butt due to the aforementioned stiffness. Outside the butt the diameter differences are still reasonable small. The shape was good for regular trees, less good for others. It must be bourne in mind that a taper spline with four measured diameters cannot give a true shape to a singular tree.

A monotony preserving quadratic spline with three measured diameters can be recommended to a taper curve only for trees of regular shape. Also in this case there exist systematic errors. The four point taper spline still gives reasonable total volumes. Therefore it can be used in volume estimation also for arbitrary trees.

Our study showed that a reduction of the number of measured diameters had a smaller effect on the monotony preserving quadratic spline than on the usual cubic spline. This is due to its structure which keeps the quadratic spline adher to its correct shape without oscillating. Therefore we can say that the fewer the number of measured diameters used the more we can recommend the use of the monotony preserving quadratic spline instead of the usual cubic spline.

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SELOSTE

Monotonisuuden säilyttävien runkokäyrien muodostamisesta

Työssä tutkittiin runkokäyrän muodostamista neliöllisen splini-funktion avulla siten, että runkokäyrä kuvaa mahdollisimman hyvin rungon kapenemisen. Perusedellytyksenä oli, että rungosta oli käytettävissä pituuden lisäksi useita mitattuja läpimittoja. Läpimittojen oletettiin antavan käsityksen rungon kapenemisesta.

Tehtävänasettelu

Kun puusta on mitattu useita läpimittoja, voidaan sille muodostaa hyvä runkokäyrä käyttäen kuutiollisia splini-funktioita (Lahtinen ja Laasasenaho 1979). Tällaisen runkokäyrän suurimpana puutteena on splini-funktion lievä heilahtelu. Heilahtelun johdosta monotonisesti kapenevan puun runkokäyrään saattaa tulla pullistumia. Ilmiö on melko yleinen. Se johtuu pääasiassa kuutiollisen splinin perusominaisuuksista, lähinnä kaarevuuden minimoinnista. Näin ollen ilmiötä ei pystytä estämään.

McAllister ja Roulier (1981a) sekä Schumaker (1983) ovat osoittaneet, että on olemassa neliöllinen splini-funktio, joka säilyttää approksimoitavan funktion monotonisuuden tietyin edellytyksin. Tältä pohjalta lähtien ryhdytään luomaan mahdollisimman hyvää algoritmia monotonisuuden säilyttävän neliöllisen splini-funktion muodostamiseksi. Tämän jälkeen tutkitaan algoritmilla saadun splini-funktion käyttöä runkokäyränä ja runkokäyrän riippuvuutta mittaustietojen lukumäärästä.

Monotonisuuden säilyttävä splini-funktio

Splini-funktiolla tarkoitetaan tässä neliöllistä tai kuutiollista polynomisplinià. Niiden ominaisuuksia ovat käsitelleet esimerkiksi Lahtinen ja Laasasenaho (1979). Muunlaisista splini-funktioista antaa tietoa esimerkiksi Schumaker (1981).

Olkoon $(x_i)_1^n$ välillä [a,b] annettu pistejoukko ja $(y_i)_1^n$ annettu reaalilukujoukko. Pistejoukon $D = ((x_i, y_i))_1^n$ sanotaan olevan kasvava (vast. vähenevä), jos $(y_i)_1^n$ on kasvava (vast. vähenevä). Vastaavasti sanotaan, että D on konveksi (vast. konkaavi) jos lukujono $(\delta_i)_1^{n-1} = ((y_{i+1} - y_i)/(x_{i+1} - x_i))_1^{n-1}$ on kasvava (vast. vähenevä). Pistejoukko D on paloittain monotoninen, jos se voidaan jakaa osiin, joista jokainen on joko kasvava tai vähenevä. Vastaavasti D on paloittain kupera, jos se voidaan jakaa osiin, joista jokainen on konveksi tai konkaavi.

Ongelmamme on nyt seuraava: Funktion arvot tunnetaan ainoastaan pistejoukossa D. On löydettävä neliöllinen splini-funktio, joka interpoloi funktiota joukossa D, ja joka on paloittain monotoninen samalla tavalla kuin D. Sanomme, että tällainen splini-funktio säilyttää monotonisuuden. Jos se on lisäksi paloittain kupera samalla tavalla kuin D, sanomme, että se säilyttää muodon.

Algoritmi

Monotonisuuden säilyttävän neliöllisen splini-funktion muodostuksen matemaattiset perustelut löytyvät artikkeleista Schumaker (1983) ja Lahtinen (1988). Itse algoritmi on esitetty tämän artikkelin kohdassa 24. Kertaamme tässä vain perusidean.

Algoritmin lähtötiedoiksi tarvitaan interpolaatiopisteiden lukumäärä n, pisteet $(x_i)_1^n$, niissä mitatut arvot $(y_i)_1^n$ sekä arvot $(m_i)_1^n$ splinin derivaatoille interpolaatiopisteissä. Näitä derivaattoja ei voi valita vapaasti, vaan niiden on toteutettava tietyt ehdot. Monotonisuuden säilyttävä neliöllinen splini konstruoidaan kullekin välille $[x_i, x_{i+1}]$ erikseen. Mikäli derivaatat m_i ja m_{i+1} eivät toteuta tiettyä ehtoa, joudutaan välille $]x_i, x_{i+1}[$ lisäämään ylimääräinen murtopiste ξ_i, jonka on sijaittava tietyllä osavälillä. Käsittelyn helpottamiseksi parametrisoidaan derivaattojen (m_i) ja ylimääräisten solmupisteiden (ξ_i) valinta. Valitsemalla derivaattaparametri a_i tietyltä väliltä A_i sekä murtopisteparametri b_i tietyltä väliltä B_i saadaan aina välille $[x_i, x_{i+1}]$ monotonisuuden säilyttävä neliöllinen splininpala. Parametrit voidaan valita myös niin, että splini-funktio säilyttää muodon, jolloin valinta on tehtävä pienemmältä väliltä.

Monotonisuuden säilyttävän splinin sopivuus runkokäyräksi

Tutkimuksen koepuuaineistona käytettiin samoja 1864 kuusen runkoa kuin Lahtinen ja Laasasenaho (1979). Kustakin puusta oli mitattu pituus sekä läpimitat 14 suhteelliselta korkeudelta. Näiden läpimittojen kautta kulkevaa Lahtisen ja Laasasenahon (1979) parasta kuutiollisella splinillä muodostettua runkokäyrää CO15 käytettiin perusrunkokäyränä, joihin monotonisuuden säilyttävän splinin avulla muodostettuja runkokäyriä verrattiin niiden ominaisuuksien selvittämiseksi.

Kaikkien mitattujen läpimittojen kautta kulkevan monotonisuuden säilyttävän neliöllisen splinin parametrit säädettiin niin, että saatu runkokäyrä QO15 antoi koepuuaineiston puille keskimääräisesti saman kokonaistilavuuden kuin CO15 (taulukko 8). Tällöin myös osatilavuudet olivat keskimäärin samat (taulukko 7). Suurin ero, 0,13%, oli korkeudella 5-10 %. Tämä ero johtuu ainakin osittain siitä, että QO15 säilyttää monotonisuuden jokaiselle puulle, mutta CO15 ei sitä aina tee. Runkokäyrä QO15 säilyttää myös muodon useimmille puille. Joissain tapauksissa on tilavuusarvion parantamiseksi luovuttu muodon säilymisestä.

Runkokäyrien QO15 ja CO15 erojen selvittämiseksi laskettiin myös niiden erotuksen keskimääräinen maksimiarvo koko rungolle ja sen osille (taulukko 10). Keskimääräinen maksimiero oli 0,5 cm, mikä sijaitsi lähes poikkeuksetta tyvellä. Ylempänä keskimääräinen maksimiero oli alle 0,1 cm.

Paremman käsityksen saamiseksi tarkasteltiin myös runkokäyrien graafisia esityksiä. Jos CO15 oli monotoninen, kulkivat runkokäyrät aivan päällekkäin lukuunottamatta tyveä, jossa oli vähäistä eroa. Jos runkokäyrässä CO15 oli pullistumia tai muita epämonotonisuuksia, syntyi luonnollisesti eroja näihin kohtiin, koska QO15 säilytti monotonisuuden (Kuvat 1,2 ja 3).

Tutkimuksista kävi selvästi ilmi, että paras monotonisuuden säilyttävän splinin avulla muodostettu runkokäyrä QO15 toistaa rungon muodon hyvin ja antaa samat kokonais- ja osatilavuudet kuin kuutiollisen splinin avulla muodostettu paras runkokäyrä CO15. Rungon apteerauksessa on QO15 selvästi parempi, koska se säilyttää monotonisuuden. Näin ollen monotonisuuden säilyttävä neliöllinen splini antaa useaa mitattua läpimittaa käytettäessä hyvän runkokäyrän, joka on käyttökelpoisempi kuin tavallisen kuutiollisen splinin antama runkokäyrä ainakin silloin kun on kysymys rungon muotoon liittyvistä asioista.

Seitsemän mitatun läpimitan käyttö

Tutkimuksen toisessa osassa selvitettiin monotonisuuden säilyttävän splinin avulla saadun runkokäyrän laatua kun käytettävien mittausten lukumäärää vähennettiin. Vertailurunkokäyränä käytettiin parasta edellä löydettyä monotonisuuden säilyttävää runkokäyrää QO15.

Ensimmäiseksi puolitettiin mittauspisteiden lukumäärä, ts. käytettiin 7 mitattua läpimittaa. Useiden kokeilujen jälkeen parhaaksi tavaksi osoittautui joka toisen mitatun läpimitan käyttö. Näiden kautta pystyttiin vetämään monotonisuuden säilyttävä neliöllinen splini niin, että saatu runkokäyrä QO8A antoi koepuuaineiston puille saman keskimääräisen kokonaistilavuuden kuin QO15. Osatilavuuksia ei enää pystytty saamaan samoiksi (taulukko 7). Eroa oli lähinnä tyvellä (0,36%), muualla erot olivat merkityksettömiä.

Runkokäyrien QO8A ja QO15 keskimääräiseksi maksimieroksi tuli 0,9 cm, mikä ero sijaitsi lähes aina tyvellä. Ylempänä rungolla keskimääräinen maksimiero oli alle 0,3 cm (taulukko 12). Tästä näkyy, että QO8A ei pysty kuvaamaan tyveä yhtä hyvin kuin QO15, mutta muualla ne kulkevat lähes samalla tavalla. Runkokäyrien graafinen tarkastelu vahvisti tätä käsitystä (kuvat 4 ja 5).

Yhteenvetona voidaan sanoa, että 7 mitatun läpimitan käyttö antaa lähes yhtä hyvän tuloksen kuin 14 mitatun läpimitan käyttö. Tyvellä syntyy hieman eroa. Taulukosta 16 käy lisäksi ilmi, että vastaavuus on parempi käytettäessä monotonisuuden säilyttävää splinià kuin käytettäessä kuutiollista splinià.

Neljän mitatun läpimitan käyttö

Seuraavaksi kokeiltiin 4 mitatun läpimitan käyttöä. Vertailurunkokäyränä oli edelleen QO15. Tällä kertaa syntyi selvempää eroa vertailurunkokäyrään kuten taulukoista 7 ja 12 käy ilmi. Keskimääräistä kokonaistilavuutta ja tyvellä olevaa osatilavuutta ei saatu yhtaikaa hyviksi, kumpikin erikseen kylläkin. Runkokäyrän keskimääräinen maksimipoikkeama vertailukäyrästä oli jo yli 1,0 cm ja tyven ulkopuolellakin 0,5 cm luokkaa. Keskimääräiset erot konstruktiossa käyttämättömiin mittaustuloksiin pysyivät kyllä edelleen pieninä (taulukot 14 ja 15). Runkokäyrien graafinen tarkastelu osoitti, että tulos on hyvä säännöllisille puille, mutta epäsäännöllistä puuta kuvattaessa ei tarkkaa vastaavuutta saada (kuvat 6 ja 7). Runkokäyrät säilyttävät kyllä aina monotonisuuden, mutta neljä läpimittaa ei välttämättä anna riittävästi tietoa rungon muodosta.

Yhteenvetona voidaan todeta että muodon säilyttävä splini antaa vielä neljän mittauspisteen tapauksessa tyydyttävän runkokäyrän. Itse asiassa runkokäyrä antaa yhtä hyviä tilavuusarvioita kuin 7 mittauspisteen kautta kulkeva kuutiollisen splini-funktion avulla muodostettu runkokäyrä.

Kolmen mitatun läpimitan käyttö

Lopuksi suoritettiin vielä muutamia kokeita 3 mitatun läpimitan käyttämisestä. Näiden kautta kulkeva monotonisuuden säilyttävä splini antaa runkokäyrän, joka on vielä tyydyttävä säännölliselle puulle (kuva 8). Jos puu on vähänkin epäsäännöllinen, ei 3 mitattua läpimittaa anna riittävästi informaatiota hyvän monotonisuuden säilyttävän runkokäyrän muodostamiseen.

Yhteenveto

Tutkimuksesta käy ilmi, että monotonisuuden säilyttävän neliöllisen splinin avulla saadaan puulle runkokäyrä, joka on ainakin yhtä hyvä kuin kuutiollisen splinin avulla muodostettu. Runkokäyrän muotoon liittyvissä asioissa on monotonisuuden säilyttävän splinin antama runkokäyrä parempi. Ero on sitä suurempi, mitä pienempää mittaustulosmäärää käytetään.

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